161. Transformation of Branching Markov Processes

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General theory for transformations of Markov processes by multiplicative functionals is well known (cf. Dynkin [1], Meyer [6], Kunita and T. Watanabe [5], and Ito and S. Watanabe [4]). Let X_t be a given Markov process and M_t be a multiplicative functional of X_t satisfying $E_x[M_t]\leqq 1$, then M_t-subprocess X_t^M of X_t is a Markov process with the transition probability $P^M(t, x, dy)=E_x[M_t\chi_{dy}(X_t)]$. When X_t is a Branching Markov process, the transformed process X_t^M happens to be not a branching Markov process. For example, if $M_t=\exp(-t)$, X_t^M is not branching Markov process. In this paper, we shall investigate necessary and sufficient conditions for M_t under which X_t^M becomes a branching Markov process, and give some examples of transformation.

1. Multiplicative functionals of branching type. Let $X_t(w)$ be a branching Markov process taking values in $\boldsymbol{S}=\bigcup_{n=0}^{\infty} S^n$, where $S^0=\{\partial\}$ and $S^\infty=\{\Delta\}$. For convenience, we assume in this paper that the process is defined on the path space W of right continuous paths and that $X_t(w)$ represents the position taken by a path $w\in W$ at time $t\geqq 0$. Precise definition of branching Markov process has been given in [2]. We shall use the terminology and the notation adopted in [2] and [3].

Let $W^{(n)}$ be the n-fold product of W and put $\tilde{W}=\bigcup_{n=0}^{\infty} W^{(n)}$, where $W^{(0)}=\{w_\partial\}$ $W^{(\infty)}=\{w_\Delta\}$.[1] We shall define a mapping ϕ of $\tilde{W}$ to W by

(1.1) $$X_t(\phi\tilde{w})=(\phi\tilde{w})(t)=\gamma\{X_t(w^1), X_t(w^2), \cdots, X_t(w^n)\},\ t\geqq 0,$$

when $\tilde{w}=(w^1, w^2, \cdots, w^n)\in W^{(n)}$, $w^j\in W$, $j=1, 2, \cdots, n$. Where γ is the mapping from $\bigcup_{n=1}^{\infty} \boldsymbol{S}^{(n)}$ to $\boldsymbol{S}$.[2]

Definition 1.1. Let M_t be an $\mathcal{N}_t$-multiplicative functional[3] of a

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1) $X_t(w_\partial)=\partial$, for all $t\geqq 0$. $X_t(w_\Delta)=\Delta$, for all $t\geqq 0$.

2) $\boldsymbol{S}^{(n)}$ is the n-fold product of $\boldsymbol{S}$. The definition of the mapping γ is given in [2].

3) Let $X=\{W, X_t, \mathcal{B}_t, P_x, x\in S\}$ be a right continuous Markov process. In this

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branching Markov process X_t satisfying a condition

(1.2) $$E_x[M_t] \leqq 1, \qquad x \in S, \qquad t \geqq 0.$$

M_t is said to be *of branching type* if it satisfies; (i) for any $n \geqq 1$,

(1.3) $$M_t(\phi\tilde{w}) = \prod_{j=1}^{n} M_t(w^j), \qquad t \geqq 0, \qquad (\text{a.s.} \tilde{P}_x,\ x \in S^n),$$

where $\tilde{w} = (w^1, w^2, \cdots, w^n) \in W^{(n)}$ and $\tilde{P}_x$ is the product measure $P_{x_1} \times \cdots \times P_{x_n}$, $(x_1, \cdots, x_n) \in x$, and (ii)

(1.4) $$P_\partial[M_t = 1] = 1, \qquad P_\Delta[M_t = 1] = 1.$$

By virtue of Theorem 1 of [2], we have

Theorem 1.1. *Let X_t be a branching Markov process,*[4] *M_t be an $\mathfrak{N}_t$-multiplicative functional satisfying* (1.2), *and let X_t^M be the M_t-subprocess of X_t. Then the following statements are equivalent;*

(i) *X_t^M is a branching Markov process,*

(ii) *M_t is a multiplicative functional of branching type.*

Taking into account the property B. III of [2], we shall weaken the condition (1.3) as follows.

Definition 1.2. Let M_t be an $\mathfrak{N}_t$-multiplicative functional of X_t subject to (1.2). M_t is said to be *of branching type in weak sense*, if it satisfies (1.4) and, for any $n \geqq 1$,

(1.3′) $$M_t(\phi\tilde{w}) = \prod_{j=1}^{n} M_t(w^j), \qquad 0 \leqq t \leqq \tau(\phi\tilde{w}), \qquad (\text{a.s. } \tilde{P}_x,\ x \in S^n),$$

where τ is the first branching time of X_t (cf. [2]).

By means of Theorem 1 of [2], we have

Theorem 1.2. *Let X_t be a branching Markov process satisfying the condition* (c.3) *of* [2], *M_t be an $\mathfrak{N}_t$-multiplicative functional of X_t subject to* (1.2), *and let X_t^M be the M_t-subprocess of X_t. If X_t^M satisfies the conditions* (c.1), (c.2), *and* (c.3) *of* [2], *then the following assertions are equivalent;*

(i) *X_t^M is a branching Markov process,*

(ii) *M_t is a multiplicative functional of branching type in weak sence.*

Remark. For wide class of subprocesses, we are able to verify the conditions (c.1), (c.2), and (c.3).

2. Examples of multiplicative funcfional of branching type.

Example 1. *Harmonic transformation.* Put $e(x) = P_x[e_\Delta = +\infty]$, ($e_\Delta$ is the explosion time). Noticing that $e(x) = \lim_{t \to \infty} T_t \hat{1}(x)$, we have

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paper, a mapping $M(t, w)$ from $[0, \infty) \times W$ to $[0, \infty)$ is said to be a multiplicative functional of X_t, if it is $\mathcal{B}_t$-measurable and if there exists a subset $W' \in \mathcal{B}_\infty$ with $P_x[W'] = 1$, $x \in S$, such that for $w \in W'$

(M.1) $M(0, w) = 1$, (M.2) $t \to M(t, w)$ is right continuous, (M.3) $M(t, w) = M(\zeta(w), w)$, $t \geqq \zeta(w)$, (M.4) $M(t+s, w) = M(t, w) \cdot M(s, \theta_t w)$. We shall call W' the *defining set* of M_t. $\mathfrak{N}_t$ is the smallest σ-field generated by X_s, $s \leqq t$.

4) When we speak about branching Markov process, it is always supposed to be a right continuous strong Markov process.

$e(\boldsymbol{x})=\hat{e}(\boldsymbol{x})$ and $T_t e(\boldsymbol{x})=e(\boldsymbol{x})$. For simplicity we assume that $0<e(\boldsymbol{x})$ for $\boldsymbol{x}\in S-\{\Delta\}$. Put

$$M_t(w)=\frac{e(X_t(w))}{e(X_0(w))}, \quad \text{if} \quad X_0(w)\neq\partial, \Delta, \tag{2.1}$$
$$=1, \qquad \text{if} \quad X_0(w)=\partial, \Delta.$$

Then M_t is a multiplicative functional of branching type. In fact,

$$M_t(\phi\tilde{w})=\frac{e(X_t(\phi\tilde{w}))}{e(X_0(\phi\tilde{w}))}=\frac{\prod_{j=1}^{n}\hat{e}(X_t(w^j))}{\prod_{j=1}^{n}\hat{e}(X_0(w^j))}=\prod_{j=1}^{n}M_t(w^j), \tag{2.2}$$

where $w=(w^1,\cdots,w^n)\in W^{(n)}$.

More generally if $e(\boldsymbol{x})=\lim_{t\to\infty}T_t\hat{f}(\boldsymbol{x})$ exists and $e(\boldsymbol{x})>0$ then (2.1) defines a multiplicative functional of branching type.

Remark. We have

$$E_{\boldsymbol{x}}[M_{t\wedge e_\Delta}]=\frac{e(\boldsymbol{x})}{e(\boldsymbol{x})},$$

therefore M_t is regular in the sense of Ito-S. Watanabe [4]. It is well known that the semi-group T_t^M of X_t^M is given by

$$T_t^M f(\boldsymbol{x})=\frac{1}{\hat{e}(\boldsymbol{x})}T_t(\hat{e}f)(\boldsymbol{x}),$$

where T_t is the semi-group of X_t.

Example 2. *Killing of the non-branching part of* X_t.

Taking a non-negative Borel measurable function f on S, and $\lambda>0$, we put

$$M_t(w)=\exp\left(-\lambda\int_0^t\check{f}(X_s(w))ds\right),^{5)} \quad \text{if} \quad X_0(w)\neq\partial, \Delta, \tag{2.3}$$
$$=1, \qquad \text{if} \quad X_0(w)=\partial, \Delta.$$

Then $M_t(w)$ is a decreasing multiplicative functional of branching type. For,

$$M_t(\phi\tilde{w})=\exp\left(-\lambda\sum_{j=1}^{n}\int_0^t\check{f}(X_s(w^j))ds\right)$$
$$=\prod_{j=1}^{n}M_t(w^j),$$

where $w=(w^1,\cdots,w^n)\in W^{(n)}$.

In particular if we take $f\equiv 1$, (2.3) reduces to

$$M_t(w)=\exp\left(-\lambda\int_0^t\xi_s(w)ds\right).^{6)}$$

In this case, X_t^M coincides with the process obtained by λ-order killing of non-branching part of X_t.

3. Construction of multiplicative functional of branching

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5) $\check{f}(x)=\begin{cases}\sum_{j=1}^{n}f(x_j), & \text{if } x\in S^n,\\ 0, & \text{if } x=\partial,\Delta.\end{cases}$

6) $\xi_t(w)=n$, if $X_t(w)\in S^n$.

type in weak sense. Let m_t be a multiplicative functional of the non-branching part of X_t. We shall construct a multiplicative functional M_t of branching type by piecing m_t together.

Classifying the path space W with respect to the number of particles at starting point, we put $W=\bigcup_{n=0}^{\infty} W_n$, where $W_n=\{w; w(0)\in S^n\}$. We define a mapping φ from the n-fold product $W_1\times \cdots \times W_1$ of W_1 to W_n by

$$(\varphi\tilde{w})(t)=\varphi[w^1(t), \cdots, w^n(t)], \tag{3.1}$$

where $\tilde{w}=(w^1, \cdots, w^n)\in W_1\times \cdots \times W_1$.

We first state

Lemma 3.1. *Let $F(w)$ be a bounded $\mathcal{N}_\infty|_{W_1}$-measurable function on W_1. If we put*

$$\tilde{F}(\varphi\tilde{w})=\prod_{j=1}^n F(w^j), \text{ for } w=(w^1, \cdots, w^n), \tag{3.2}$$

then $\tilde{F}$ is well-defined as a bounded $\mathcal{N}_\infty|_{W_n}$-measurable function on W_n.

Now, let $X^0=\{W_1, \mathcal{N}_t|_{W_1}, X_t, t<\tau, P_x, x\in S\}$ be the non-branching part on S of X_t, and m_t be an $\mathcal{N}_t|_{W_1}$-multiplicative functional of X^0 whose defining set is W_1'.

For $n\geqq 0$, we extend m_t onto W_n as follows; when $n\geqq 1$, we put

$$\begin{aligned}\tilde{m}_t(\varphi\tilde{w})&=\prod_{j=1}^n m_t(w^j), \quad \text{if} \quad t\leqq\tau(\varphi\tilde{w}),\\ &=\tilde{m}_{\tau(\varphi\tilde{w})}(\varphi\tilde{w}), \quad \text{if} \quad t\geqq\tau(\varphi\tilde{w}),\end{aligned} \tag{3.3}$$

and when $n=0$, we put

$$\tilde{m}_t(\varphi\tilde{w})=1.$$

Then, $\tilde{m}_t$ is well-defined by the above Lemma. It is easy to see that the defining set of $\tilde{m}_t$ is $W_n'=\varphi(W_1'\times \cdots \times W_1')$.

Thus, we have a functional $\tilde{m}_t$ on $W=\bigcup_{n=0}^{\infty} W_n$ with the defining set $W'=\bigcup_{n=0}^{\infty} W_n'$.

Now, let us define $M_t(w)$ as follows:

$$M_t(w)=\tilde{m}_\tau(w)\theta_{\tau_1}\tilde{m}_\tau(w)\theta_{\tau_2}\tilde{m}_\tau(w)\cdots\theta_{\tau_{j-1}}\tilde{m}_\tau(w)\tilde{m}_{t-\tau_j(w)}(\theta_{\tau_j}w), \text{ on } A_j^t, \tag{3.4}$$

where

$$\begin{aligned}&\theta_{\tau_j}\tilde{m}_\tau(w)=\tilde{m}_{\tau(\theta_{\tau_j}w)}(\theta_{\tau_j}w), \quad \text{and}\\ &A_j^t=\{w; \tau_j(w)\leqq t<\tau_{j+1}(w)\}.^{7)}\end{aligned} \tag{3.5}$$

Then M_t has the following properties:

Lemma 3.2. *$M_t(w)$ is $\mathcal{N}_{t+}$-measurable.*[8]

Lemma 3.3. *$M_t(w)$ has the multiplicativity*, i.e.

7) $\tau_0=0,\ \tau_1=\tau,\ \tau_n=\tau_{n-1}+\tau(\theta_{\tau_{n-1}}w)$.

8) $\mathcal{N}_{t+}=\bigcap_{n>} \mathcal{N}_{t+\frac{1}{n}}$.

(3.6) $$M_{t+s}(w)=M_t(w)M_s(\theta_t w),\quad w\in W'.$$

Lemma 3.4. *If $m_t(w)\leqq 1$, then $M_t(w)\leqq 1$, $w\in W'$.*

Lemma 3.5. *If M_t satisfies $E_x[M_\tau]=1$, then*

(3.7) $$E_x[M_{t\wedge\tau_n}]=1,$$

for any n. Therefore we have

$$E_x[M_t]\leqq 1.$$

If $\{M_{t\wedge\tau_n};\ n=0,1,2,\cdots\}$ is uniformly integrable, we have

$$E_x[M_t]=1.$$

Combining the above Lemmas, we have

Theorem 3.1. *Let $X^0=\{W_1, \mathcal{N}_t|_{W_1}, X_t, t<\tau, P_x, x\in S\}$ be the non-branching part on S of a branching Markov process X_t, and let m_t be a multiplicative functional of X^0 satisfying either*

(i) $m_t\leqq 1$, *or*

(ii) $E_x[m_\tau]=1$, $x\in S$.

Then, there exists an $\mathcal{N}_{t+}$-multiplicative functional M_t of branching type in weak sense such that

(3.8) $$M_t(w)=m_t(w),\quad w\in W_1,\quad t\leqq\tau(w),$$

and

(i′) $M_t\leqq 1$, *or*

(ii′) $E_x[M_t]\leqq 1$, $x\in S$,

respectively. If $\{M_{t\wedge\tau_n};\ n=0,1,2,\cdots\}$ is uniformly integrable, (ii′) *is replaced by*

(ii″) $E_x[M_t]=1$.

4. Transformation of drift. Let $\tilde{X}=(\tilde{X}_t, \tilde{\mathscr{B}}_t, \tilde{P}_x)$ be a Hunt process satisfying the condition (L) of Meyer [8]. The following facts are known (cf. [9], [10]): Let B_t^k, $k=1,2,\cdots,d$, be continuous additive functionals[9] of X satisfying

$$\tilde{E}_x[B_t^k]=0,\quad\text{and}\quad \tilde{E}_x[(B_t^k)^2]<+\infty.$$

For $f_k\in L^2(B^k)$, if we put

(4.1) $$m_t=\exp\left(\sum_{k=1}^d f_k\cdot B_t^k-\frac{1}{2}\sum_{k=1}^d |f_k|^2\cdot\langle B^k\rangle_t\right),$$

then, it is a multiplicative functional of $\tilde{X}$ which satisfies the following[10]

Lemma 4.1. *Let σ be a finite valued Markov time of $\tilde{X}$ satisfying*

(4.2) $$\{t<\sigma\}\subset\{\sigma\leqq t+\sigma(\theta_t w)\}.$$

If $\sup_{x\in S}\tilde{E}_x\left[\sum_{k=1}^d |f_k|^2\cdot\langle B\rangle_\sigma\right]<+\infty$, then $\tilde{E}_x[m_\sigma]=1$.

Now we assume that the non-branching part X^0 of a branching

9) If $\exp(-A_t)$ is a multiplicative functional, A_t is said to be an additive functional.

10) The exact statements are found in [7] and [8]. In particular, as for the definition of the stochastic integral $f\cdot B$, see [7] or [8].

Markov process is equivalent to the $\exp(-\varphi_t)$-subprocess $\dot{X}_t$ of $\tilde{X}=(\tilde{X}_t, \tilde{\mathscr{B}}_t, \tilde{P}_x)$ subject to the condition (L) of Meyer, where φ_t is a non-negative additive functional of X_t. Then X_t^0 is obtained by killing $\tilde{X}_t$ at a random time σ and by enlarging $\tilde{\mathscr{B}}_t$ sufficiently large, if necessary, we can assume σ is a $\tilde{\mathscr{B}}_t$ Markov time for which (4.2) is easily verified. We assume $\tilde{E}_x(m_\sigma)=1$. Now $m_{t\wedge\sigma}$ can be identified with a multiplicative functional of X_t^0.

Definition 4.1. The multiplicative functional M_t defined by (3.4), using the multiplicative functional $m_{t\wedge\tau}$ of $\dot{X}_t$, where m_t is defined by (4.1), is said to be a *multiplicative functional of drift.*

Example 3. *Transformation of drift for d-dimensional branching Brownian motion.*

Now, we consider a branching Markov process X_t whose non-branching part is the $\exp\left(-\int_0^t c(x_s)ds\right)$-subprocess of d-dimensional Brownian motion x_t, where $c(x)$ is a bounded continuous function on $S=R^d$ satisfying $c(x)\geqq c>0$.

Put

$$B_t^k = x_t^{(k)} - x_0^{(k)}, \quad x_t=(x_t^{(1)}, x_t^{(2)}, \cdots, x_t^{(d)}), \tag{4.3}$$

take bounded continuous functions b_k on S, $k=1, 2, \cdots, d$, and define

$$m_t = \exp\left(\sum_{k=1}^d b_k \cdot B_t^k - \frac{1}{2}\sum_{k=1}^d \int_0^t |b_k(x_s)|^2 ds\right). \tag{4.4}$$

Then the condition of Lemma 4.1 is satisfied for the killing time and so we can construct the multiplicative functional of drift which is of branching type in weak sense.

Let X_t^M be the M_t-subprocess of the branching Markov process X_t, then the backward equation of X_t^M is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sum_{k=1}^d b_k(x)\frac{\partial u}{\partial x_k} + c(x)(F[x; u]-u), \tag{4.5}$$

while the backward equation of X_t is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + c(x)(F[x; u]-u). \tag{4.6}$$

Thus M_t induces a drift.

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