152. *The Lattice of Congruences of Locally Cyclic Semigroups*

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In [2] Dean and Oehmke proved Theorem 1. Using Theorem 2 proved by Tamura and Levin [4] we will give another proof for Theorem 1.

Theorem 1. *The lattice of congruences on a locally cyclic semigroup is a distributive lattice.*

Theorem 2. *Let S be a locally cyclic semigroup, then $S=\bigcup_{i=1}^{\infty} S_i$ where $S_i \subseteq S_{i+1}$ and S_i is a cyclic semigroup.*

Let C be a cyclic semigroup. Denote C by $C=(n, m)$ where 1 generates C and n, m are non-negative integers or $n=m=\infty$. C is finite if and only if n, m are finite. See p. 19–20 [1].

Any congruence ρ on a cyclic semigroup C is determined uniquely by its induced homomorphic image C' a cyclic semigroup. We denote $\rho=\rho(n', m')$ where $C'=(n', m')$ and

$$(1) \qquad a\rho b \text{ if and only if } \begin{cases} a=b & a<n',\ b<n' \\ m' \mid (a-b) & a \geq n',\ b \geq n'. \end{cases}$$

Proposition 1. *Let $C=(n, m)$ be a cyclic semigroup $\rho=\rho(n_1, m_1)$ is a congruence on C if and only if $n_1 \leq n$, $m_1 \mid m$.*

Proposition 2. *Let* S_1, S_2 be cyclic semigroups such that $S_1 \subseteq S_2$ and 1 generates S_2, k generates S_1. $\rho_1=\rho_1(n_1, m_1)$ and $\rho_2=\rho_2(n_2, m_2)$ are congruences on S_1 and S_2 respectively with $\rho_1=\rho_2 \mid S_1$ if and only if $n_2 \leq n_1$ and $n_1-r \leq n_2-1$ where $n_1 \equiv r (\text{mod}\, k)$, $1 \leq r \leq k$, and $m=\text{lcm}\,(k, m_2)$.

Definition 1. Let σ, ρ be congruences on a groupoid G. Then $\sigma \vee \rho$ is the smallest congruence containing σ and ρ and $\sigma \wedge \rho$ is the largest congruence contained in σ and ρ.

Since the identity relation is contained in all congruences and the universal relation contains all congruences and intersection preserves congruences for any congruences, σ, ρ on a groupoid G both $\sigma \vee \rho$ and $\sigma \wedge \rho$ exist.

In [5] Tamura proved the following.

Proposition 3. *Let C be a cyclic semigroup; let $\sigma=\sigma(n_1, m_1)$, $\rho=\rho(n_2, m_2)$ be congruences on C then*

(i) $\sigma \vee \rho=(\min\,(n_1, n_2), \gcd\,(m_1, m_2))$

(ii) $\sigma \wedge p=(\max\,(n_1, n_2), \text{lcm}\,(m_1, m_2))$.

As a consequence of Proposition 3 we have:

Proposition 4. *Let σ, ρ, δ be congruences on a cyclic semigroup C. Then* $\sigma\wedge(\rho\vee\delta)=(\sigma\wedge\rho)\vee(\sigma\wedge\delta)$.

Definition 2. Let S be a locally cyclic semigroup and σ a congruence on S. Then $\sigma_i=\sigma\,|\,S_i$ where $S=\bigcup_{i=1}^{\infty}S_i$ and S_i is a cyclic semigroup.

Since the representation of S is not unique, σ_i depends upon the S_i's.

Proposition 5. *Let S, σ be as defined above. Then*

(i) σ_i *is a congruence,* $1\leq i\leq\infty$

(ii) $\sigma_i\subseteq\sigma_{i+j}$, $0\leq j<\infty$

(iii) $\sigma_i=\sigma_{i+j}\,|\,S_i$, $0\leq j<\infty$

(iv) $\sigma=\bigcup_{i=1}^{\infty}\sigma_i$.

By [3] we have the following two propositions.

Proposition 6. *Let σ, ρ be congruences on a groupoid G. Then $\sigma\vee\rho=(\sigma\cup\rho)T$ where $T=\bigcup_{n=1}^{\infty}T_2^n$, $(\delta)T_2=\delta\cup\delta^2$, $\delta T_2^n=((\delta)T_2)T^{n-1}$, and "$\cup$" is the set union.* (See [3].)

Proposition 7. *Let $\delta\subseteq G\times G$ for some groupoid G, and $a, b\in G$. Then $a(\delta)Tb$ if and only if there exists $x_1, \cdots, x_n\in G$ such that* $a=x_1\delta x_2$, $x_2\delta x_3$, $\cdots$, $x_{n-1}\delta x_n=b$.

Proposition 8. *Let S be a locally cyclic semigroup with congruences σ, δ and let $S=\bigcup_{i=1}^{\infty}S_i$, S_i a cyclic semigroup. Then*

(i) $\sigma_i\vee\rho_i=(\sigma\vee\rho)_i$

(ii) $\sigma_i\wedge\rho_i=(\sigma\wedge\rho)_i$.

We will prove only (i) since the proof of (ii) is an obvious result of the definition of "$\wedge$".

Clearly $\sigma_i\vee\rho_i\subseteq(\sigma\vee\rho)_i$; therefore assume $a, b\in S_i$ and $a(\sigma\vee\rho)_i b$ and $a\neq b$. Since $\sigma_i\vee\rho_i$ is symmetric without loss of generality assume $a<b$. By Proposition 6 $a(\sigma\vee\rho)Tb$ so by Proposition 7 there exists $x_1, \cdots, x_n$ such that $a=x_1(\sigma\vee\rho)x_2, \cdots, x_{n-1}(\sigma\vee\rho)x_n=b$ with $x_j\in S_{ij}$, $1\leq j\leq n$. Let $i*=\max[\{i_j\}\cup\{i\}]$. We have $x_1, \cdots, x_n\in S_{i*}$ since $S_i\subseteq S_{i*}$ and $S_{ij}\subseteq S_{i*}$ $1\leq j\leq n$, and $x_j(\sigma\vee\rho)x_{j+1}$ implies $x_j\sigma_{i*}x_{j+1}$ or $x_j\rho_{i*}x_{j+1}$. Let $\sigma_{i*}=\sigma_{i*}(\bar{n}*, \bar{m}*)$ and $\rho_{i*}=\rho_{i*}(n*, m*)$. Using (1) we have $\bar{m}*\,|\,x_j-x_{j+1}$ or $m*\,|\,x_j-x_{j+1}$ so $\gcd(\bar{m}*, m*)\,|\,x_i-x_{i+1}$ giving us

(2) $$\gcd(\bar{m}*, m*)\,|\,a-b \quad\text{since}\quad a-b=\sum_{j=1}^{n-1}(x_j-x_{j+1}).$$

Now since $a, b\in S_i$, $k\,|\,a-b$ where k generates S_i as a subsemigroup of S_{i*}. Therefore by (2) $\operatorname{lcm}(k, \gcd(\bar{m}*, m*))\,|\,a-b$. But $\operatorname{lcm}(k, \gcd(\bar{m}*, m*))=\gcd(\operatorname{lcm}(k, \bar{m}*), \operatorname{lcm}(k, m*))=\gcd(\bar{m}_i, m_i)$ where $\sigma_i=(\bar{n}_i, \bar{m}_i)$ and $\rho_i=(n_i, m_i)$ by Proposition 2. This gives

(3) $$\gcd(\bar{m}_i, m_i)\,|\,a-b.$$

By Proposition 2 and (1) either $\bar{n}_i - \bar{r} \leq \bar{n}_{i*} - 1 < a$ or $n_i - r \leq n_{i*} - 1 < a$ since $a \neq b$. Now $k \mid \bar{n}_i - \bar{r}$, $k \mid n_i - r$, and $k \mid a$ so $\bar{n}_i \leq a$ or $n_i \leq a$ since $1 \leq \bar{r} \leq k$ and $1 \leq r \leq k$ therefore

$$\min(\bar{n}_i, n_i) \leq a < b. \tag{4}$$

From Proposition 3, $\sigma_i \vee \rho_i = (\min(\bar{n}_i, n_i), \gcd(\bar{m}_i, m_i))$ so (3) and (4) give us

$$a(\sigma_i \vee \rho_i)b. \tag{5}$$

Therefore $(\sigma \vee \rho)_i \subseteq \sigma_i \vee \rho_i$ which gives

$$\sigma_i \vee \rho_i = (\sigma \vee \rho)_i.$$

Now using Theorem 2 and Propositions 4, 5, and 8 we will give another proof for Theorem 1.

Theorem 1. *Let S be a locally cyclic semigroup and $\mathcal{S}$ the lattice of congruences on S. Then $\mathcal{S}$ is a distributive lattice.*

Let $\sigma, \rho, \delta \in \mathcal{S}$. By Theorem 2 and Proposition 5, $S = \bigcup_{i=1}^{\infty} S_i$, $S_i \subseteq S_{i+1}$, S_i a cyclic semigroup $1 \leq i < \infty$ and σ_i, ρ_i δ_i are all well defined congruences with respect to $\{S_i\}_{1 \leq i < \infty}$ for $1 \leq i < \infty$.

Therefore
$$\begin{aligned}
\sigma \wedge (\rho \vee \delta) &= \bigcup_{i=1}^{\infty} [\sigma \wedge (\rho \vee \delta)]_i = \bigcup_{i=1}^{\infty} [\sigma_i \wedge (\rho \vee \delta)_i] \\
&\qquad \text{by Prop. 5} \qquad \text{by Prop. 8} \\
&= \bigcup_{i=1}^{\infty} [\sigma_i \wedge (\rho_i \vee \delta_i)] = \bigcup_{i=1}^{\infty} [(\sigma_i \wedge \rho_i) \vee (\sigma_i \wedge \delta_i)] \\
&\qquad \text{by Prop. 8} \qquad \text{by Prop. 4} \\
&= \bigcup_{i=1}^{\infty} [(\sigma \wedge \rho)_i \vee (\sigma \wedge \delta)_i] = \bigcup_{i=1}^{\infty} [(\sigma \wedge \rho) \vee (\sigma \wedge \delta)]_i \\
&\qquad \text{by Prop. 8} \qquad \text{by Prop. 8} \\
&= (\sigma \wedge \rho) \vee (\sigma \wedge \delta) \\
&\qquad \text{by Prop. 5.}
\end{aligned}$$

References

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