196. A Probabilistic Treatment of Semi-Linear Parabolic Equations

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Recently N. Ikeda, M. Nagasawa, and S. Watanabe have given a definition of branching Markov processes in general set up and have gotten several results about their structure [1], [2], and [3]. The purpose of this paper is to extend their methods in order to give a probabilistic treatment for semi-linear parabolic equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + F(u)$ which was discussed by A. Kolmogoroff, I. Petrovsky, and N. Piscounoff [5] (abbreviated as KPP-equation). When we deal with KPP-equation, one of the difficulties comes from the fact that some coefficients of $F(u)$ may be negative even when $F(u)$ is a polynomial. In general, it happens that the solution of a semi-linear parabolic equation takes negative values and infinite values even for a positive bounded initial value. So if we want to treat it in probabilistic way, we must introduce some artificial procedure. One possible way is perhaps to permit the fundamental probability measure of the process to take signed values and infinite total mass. But we do not take this way. In this paper it is solved by introducing two kinds of technical operation, but it will be seen that they have natural intuitive probabilistic interpretation. One of them is to extend the state space of the processes in appropriate way, and the other is to make an operation to the initial value (cf (1.1) and (3.2)).

1. Notations and definitions. Following [2], we introduce some notations. Let R_d be d-dimensional Euclidian space, $\bar{R}_d$ be the one-point compactification of R_d, and let $N=\{0, 1, 2, 3, \cdots\}$. Also, let $S=\bar{R}_d\times N$ be the topological sum of $\bar{R}_d\times\{i\}$, $i\in N$. We denote the n-fold product of S with itself by $S^{(n)}$ and we say that $\boldsymbol{z}=((x_1, k_1), (x_2, k_2), \cdots, (x_n, k_n))\in S^{(n)}$ is R-equivalent to $\boldsymbol{z}'=((x_1', k_1'), (x_2', k_2'), \cdots, (x_n', k_n'))\in S^{(n)}$, if $(x_1, x_2, \cdots, x_n)$ is obtainable by a permutation of $(x_1', x_2', \cdots, x_n')$ and if $k_1+k_2+\cdots+k_n=k_1'+k_2'\cdots+k_n'$. Let us denote the quotient spaces $S^{(n)}/R$ by S^n, and write $\boldsymbol{z}=(\boldsymbol{x}, \boldsymbol{k})$ if $\boldsymbol{x}=(x_1, x_2, \cdots, x_n)$ and $\boldsymbol{k}=(k_1, k_2, \cdots, k_n)$. In the following, we write as $|\boldsymbol{k}|=k_1+k_2+\cdots+k_n$.

Now, $\bar{R}_d^n=\{\boldsymbol{x}; \boldsymbol{z}=(\boldsymbol{x}, \boldsymbol{k})\in S^n\}$ and S^n are metric spaces. Let us consider the topological sum $\bigcup_{n=0}^{\infty} S^n$, where $S^0=\{\partial\}\cup N$, ∂ being an extra point. Clearly $\bigcup_{n=0}^{\infty} S^n$ is isomorphic to $\bigcup_{n=0}^{\infty} \bar{R}_d^n\times N$. Let $\boldsymbol{S}=\bigcup_{n=0}^{\infty} S^n\cup\{\Delta\}$ be the one-point compactification of $\bigcup_{n=0}^{\infty} S^n$, and $\boldsymbol{C}(\boldsymbol{S})$ be the space of bounded continuous functions on $\boldsymbol{S}$. We also use the space $\bar{\boldsymbol{C}}^*(\bar{R}_d)$ of all continuous functions on $\bar{R}_d$ with $\|f\|\leqq 1$, and the space $\boldsymbol{B}(\bar{R}_d)$ of measurable functions on $\bar{R}_d$.

For non-negative λ, we define a mapping $\wedge$ from $f\in\boldsymbol{B}(\bar{R}_d)$ to the space of measurable functions on $\bigcup_{n=0}^{\infty} S^n$ by

$$\widehat{f\cdot\lambda}(z)=\begin{cases}\lambda^{|k|}, & \text{if } z=(\partial, k)\in S^0\\ \lambda^{|\boldsymbol{k}|}f(x_1)f(x_2)\cdots f(x_n) & \text{if } z=(\boldsymbol{x},\boldsymbol{k})\in S^n\end{cases}\tag{1.1}$$

where $\boldsymbol{x}=(x_1, x_2, \cdots, x_n)$ and $\boldsymbol{k}=(k_1, k_2, \cdots, k_n)$. For the convenience of later use, we put $\widehat{f\cdot\lambda}(\Delta)=0$.

For $\lambda\leqq 1$, $\widehat{f\cdot\lambda}\in\boldsymbol{C}(\boldsymbol{S})$, but when $\lambda>1$, $\widehat{f\cdot\lambda}$ is an unbounded function.

Definition 1.1. A right continuous strong Markov process $\{Y_t=(X_t, K_t), \zeta, \mathcal{B}_t, P_z; z\in\boldsymbol{S}\}$ on $\boldsymbol{S}$ is said to be a *branching Markov process with age*, if its semi-group $\{T_t; t\geqq 0\}$ satisfies

$$T_t(\widehat{f\cdot\lambda})=\widehat{(T_t(\widehat{f\cdot\lambda}))|_{\bar{R}_d}\cdot\lambda}, \text{ for } f\in\bar{\boldsymbol{C}}^*(\bar{R}_d),^{1)}\tag{1.2}$$

where $\lambda\geqq 0$ and $t\geqq 0$ vary over the values to which the both side of (1.2) have definite values.

Clearly if $\lambda\leqq 1$, they are bounded for all $t\geqq 0$. We remark also that if we impose always $\lambda=1$, the above mentioned process reduces to a branching Markov process treated in [2]. Intuitively, the process $Y_t=(X_t, K_t)$ may be considered as follows: X_t denotes the position of n-particles, while K_t denotes the ages of the particles, but we are only interested in the total of ages.

Now we put

$$\xi_t(w)=n, \quad \text{if } Y_t(w)\in S^n,$$

and define the following Markov times:[2]

$\tau(w)=\inf\{t; \xi_t(w)\neq\xi_0(w)\}$, $(\inf\phi=\infty)$,

$\sigma(w)=\inf\{t<\tau; |K_t(w)|\neq|K_0(w)|\}$, $(\inf\phi=\infty)$,

$\tau_0(w)=0$, $\tau_1(w)=\tau(w)$, and $\tau_n(w)=\tau_{n-1}(w)+\theta_{\tau_{n-1}}\tau(w)$, $(n\geqq 2)$,

$\sigma_0(w)=0$, $\sigma_1(w)=\sigma(w)$, and $\sigma_n(w)=\sigma_{n-1}(w)+\theta_{\sigma_{n-1}}\sigma(w)$, $(n\geqq 2)$.

τ is called the first branching time.

2. S-equation 1. Let $\{Y_t=(X_t, K_t), \zeta, \mathcal{B}_t, P_z; z\in\boldsymbol{S}\}$ be a branching Markov process with age on $\boldsymbol{S}$. We assume that Y_t satisfies the following condition.

Condition 1: (i) It holds that

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1) For $f(\boldsymbol{x},\boldsymbol{k})$ defined on $\boldsymbol{S}$, $f|_{\bar{R}_d}(x)=f|_{\bar{R}_d\times\{0\}}(x, 0)$.

2) A non-negative random variable is said to be a Markov time if $\{w; \tau(w)<t\}\in\mathcal{B}_t$ for any $t\geqq 0$. Cf. eg. [6].

$P_{(x,k)}[X_t \in ,B \mid |K_t|=k+p]=P_{(x,0)}[X_t \in B, \mid |K_t|=p], (x, k) \in S,$
for any Borel set $B \subset \bigcup_{n=0}^{\infty} \bar{R}_d^n$ and $p \in N$.

(ii) $|K_t|$ is increasing in t, and for any $A \in \mathscr{B}_{\sigma_n}$,[3] it holds that

$$P_{(x,0)}[A, |K_{\sigma_n}|=|K_{\sigma_{n-1}}|+1, \sigma_n<\tau]=P_{(x,0)}[A, \sigma_n<\tau], x \in \bar{R}_d.$$

(iii) There exist a positive bounded function $k(x)$ and a system $\{q_n(x); n=0, 2, 3, 4, \cdots\}$ of non-negative Borel measurable functions on $\bar{R}_d$ such that $\sum_{n\neq 1} q_n(x)=k(x)$ and that for any Borel set $B \subset S$

$$P_{(x,0)}(Y_\tau \in B \mid \tau, Y_{\tau-}=(a, p))=q_n(a)\delta_{(\boldsymbol{a},\boldsymbol{p})}(B)/k(a), x \in \bar{R}_d,$$

where $\boldsymbol{a}=(a, a, \cdots, a) \in \bar{R}_d^n$, and $\boldsymbol{p}=(p_1, p_2, \cdots, p_n)$ with $|\boldsymbol{p}|=p$. Moreover, for any Borel set $B \subset S$,

$$P_{(x,0)}(X_\sigma \in B, K_\sigma=p \mid \sigma, Y_{\sigma-})=\delta_{(X_{\sigma-}, K_{\sigma-}+1)}(B, p), x \in R_d.$$

(iv) For any Borel set $BC\bar{R}_d$ and $(x, k) \in S$, it holds that

$$P_{(x,k)}[X_{\tau-} \in B, \tau \in dt, \sigma_p \leqq t<\sigma_{p+1}] = E_{(x,k)}\left[\chi_B(X_t)e^{-2\int_0^t k(X_s)ds} \frac{\left(\int_0^t k(X_s)ds\right)^p}{p!} K(X_t)dt \right],$$

and

$$P_{(x,k)}[X_{\sigma-} \in B, \sigma \in dt]=E_{(x,k)}\left[\chi_B(X_t)e^{-2\int_0^t k(X_s)ds}k(X_t)dt\right].$$

Now, let us consider the process Y_t^0 on S which is obtained by killing Y_t at the first branching time τ and restricted on S, and denote the probability measure of Y_t^0 by $P_{(x,k)}^0$. The integration by $P_{(x,k)}^0$, and the semigroup of Y_t^0 are denoted as $E_{(x,k)}^0$ and T_t^0, respectively, i.e. for any $f \in \boldsymbol{B}(\bar{R}_d)$,

(2.1) $$T_t^0\widehat{f\cdot\lambda}(x, k)=E_{(x,k)}^0[\widehat{f\cdot\lambda}(Y_t^0)]=E_{(x,k)}[\widehat{f\cdot\lambda}(Y_t); t<\tau].$$

Theorem 2.1. *Let Y_t be a branching Markov process with age on S satisfying the condition 1. If $u(t, x)=T_t\widehat{f\cdot\lambda}(x, 0)$ exists for $x \in \bar{R}_d$ and $t \geqq 0$, then $u(t, x)$ is the solution, with the initial value f, of the integral equation (S-equation)*

(2.2) $$u(t, x)=T_t^0\widehat{f\cdot\lambda}(x, 0)+\int_0^t\int_S K((x, 0), ds, d(y, p))\lambda^p F(y, u(t-s, y)),$$

where K is defined by

(2.3) $$K((x, 0), dt, B)=P_{(x,0)}[\tau \in dt, Y_{\tau-} \in B],$$

for any $x \in \bar{R}_d$ and any Borel subset B of S, and F is defined by

(2.4) $$F(x, u)=\frac{1}{k(x)}\sum_{n\neq 1} q_n(x)u^n.$$

Proof is obtained by using Dynkin's formula [6] and the condition 1.

3. S-equation 2. It must be noticed that in (2.4), q_n is non-negative and $n=1$ is omitted in the summation. In this section we shall exclude this limitation. Let S_1 be the product space $S \times J$ of S and $J=\{0, 1, 2, 3\}$.

3) For any Markov time τ, $\mathscr{B}_\tau$ denotes the σ-algebra generated by the set A such that $A \in \mathscr{B}_\infty$ and, for any t, $A \cap \{\tau<t\} \in \mathscr{B}_t$.

Now, taking $\lambda \geqq 0$, we define a mapping $\sim$ from $f \in \boldsymbol{B}(\bar{R}_d)$ to the space of measurable functions on $\boldsymbol{S} \times \boldsymbol{J}$ by

$$(3.1) \qquad \widetilde{f\cdot\lambda}(\boldsymbol{x}, \boldsymbol{k}, j) = (-1)^{[\frac{j}{2}]} \widehat{f\cdot\lambda}(\boldsymbol{x}, \boldsymbol{k}),$$

where $(\boldsymbol{x}, \boldsymbol{k}, j) \in \boldsymbol{S} \times \boldsymbol{J}$.

Definition 3.1. A right continuous strong Markov process $\{Z_t = (X_t, K_t, J_t), \zeta, \mathscr{B}_t, P_{(z,j)}; (z, j) \in \boldsymbol{S} \times \boldsymbol{J}\}$ is said to be a *signed branching Markov process* (with age), if its semi-group $\{U_t; t \geqq 0\}$ satisfies

$$(3.2) \qquad U_t \widetilde{f\cdot\lambda} = \widehat{(U_t \widetilde{f\cdot\lambda}) |_{\bar{R}_d} \cdot \lambda},^{4)} \text{ for } f \in \boldsymbol{C}^{\overline{*}}(\bar{R}_d),$$

where $\lambda \geqq 0$ and $t \geqq 0$ vary over the values to which the both sides of (3.2) have definite values.

We define the first branching time η of Z_t by

$$\eta(w) = \inf\{t; J_t(w) \neq J_0(w)\} \wedge \tau(w),$$

where τ is the first branching time of $Y_t = (X_t, K_t)$, and put

$$\eta_0(w) = 0,\ \eta_1(w) = \eta(w), \text{ and } \eta_n(w) = \eta_{n-1}(w) + \theta_{\eta_{n-1}} \eta(w),\ (n \geqq 2).$$

Now, we assume that Z_t satisfies the following condition.

Condition 2. (i) There exist a positive bounded function $k(x)$ and a system $\{(q_n^+(x), q_n^-(x)); n = 0, 1, 2, 3, \cdots\}$ of non-negative bounded Borel measurable functions on $\bar{R}_d$ such that $q_n^+(x) q_n^-(x) = 0$ and $\sum_{n=0}^{\infty} (q_n^+(x) + q_n^-(x)) = k(x)$.

(ii) If we write, for short,

$$P_{(x,k,j)}[Y_\eta \in S^n, J_\eta = j' \mid Z_{\eta-}] = [j, j']_n, \text{ for } (x, k) \in S,$$

then it holds that

$$[0, 1]_n = [1, 0]_n = [2, 3]_n = [3, 2]_n = q_n^+(X_{\eta-})/k(X_{\eta-}), \text{ and}$$
$$[0, 3]_n = [1, 2]_n = [2, 1]_n = [3, 0]_n = q_n^-(X_{\eta-})/k(X_{\eta-}).$$

(iii) For any fixed $j \in \boldsymbol{J}$, $\{Y_t = (X_t, K_t), \mathscr{B}_t, P_{(z,j)}; \boldsymbol{z} \in \boldsymbol{S}\}$ is, independent of j, a branching Markov process with age on $\boldsymbol{S}$ (cf. Def. 1.1), and it satisfies the condition 1, if we put $q_n(x) = q_n^+(x) + q_n^-(x)$ and replace τ by η in condition 1, (iii) (the case $n=1$ must be included).

The above defined signed branching Markov process has an intuitive interpritation as follows: $\boldsymbol{S} \times \{0\}$ and $\boldsymbol{S} \times \{1\}$ represent a positive world, while $\boldsymbol{S} \times \{2\}$ and $\boldsymbol{S} \times \{3\}$ do a negative (or anti) an world, and a particle travels both worlds bringing the effects of these world subject to Markov property.

Now, we state

Theorem 3.1. *Let Z_t be a signed branching Markov process on $\boldsymbol{S} \times \boldsymbol{J}$ satisfying the condition 2. If $u(t, x) = U_t \widetilde{f\cdot\lambda}(x, 0, 0)$ exists for $x \in \bar{R}_d$ and $t \geqq 0$, then $u(t, x)$ is the solution, with the initial value*

4) For $f(\boldsymbol{x}, \boldsymbol{k}, j)$ defined on $\boldsymbol{S} \times \boldsymbol{J}$, $f|_{\bar{R}_d}(x) = f|_{\bar{R}_d \times \{0\} \times \{0\}}(x, 0, 0)$.

$f\in\bar{C}^*(\bar{R}_d)$, *of the integral equation (S-equation)*

$$u(t,x)=U_t^0\widetilde{f\cdot\lambda}(x,0,0)+\int_0^t\int_S K((x,0,0),ds,d(y,p,0))\lambda^p F(y,u(t-s,y)), \tag{3.3}$$

where U_t^0, K, *and* F *are defined by*

$$U_t^0\widetilde{f\cdot\lambda}(x,0,0)=E_{(x,0,0)}[\widetilde{f\cdot\lambda}(Z_t);\ t<\eta], \tag{3.4}$$

$$K((x,0,0),ds,d(y,p,0))=P_{(x,0,0)}[\eta\in ds,\ Y_{\eta-}\in d(y,p)],\ \textit{and} \tag{3.5}$$

$$k(x)F(x,u)=\sum_{n=0}^{\infty}(q_n^+(x)-q_n^-(x))u^n. \tag{3.6}$$

Proof is obtained by Dynkin's formula and the condition 2.

4. A probabilistic approach to KPP-equation. A. Kolmogoroff, I. Petrovsky, and N. Piscounoff [5] discussed a parabolic equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+F(u),^{5)} \tag{4.1}$$

where F satisfies the following conditions:

$$\begin{aligned}&F(0)=F(1)=0,\ F(v)>0\ (0<v<1),\ \text{and}\\ &F'(0)=1>F'(v)\ (0<v\leqq 1).\end{aligned} \tag{4.2}$$

As is well known, the solution of (4.1), with initial value f, is given by

$$u(t,x)=\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{(y-x)^2}{2t}}f(y)dy+\int_0^t ds\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi s}}e^{-\frac{(y-x)^2}{2s}}F(u(t-s,y))dy. \tag{4.3}$$

Now let us denote the killed process of Z_t at the time $\eta\wedge\sigma$ by $Z_t'=(X_t',K_t',J_t')$ and assume that X_t' is the $\exp\left(-2\int_0^t k(B_s)ds\right)$-subprocess of standard Brownian motion B_t. Moreover we assume that F defined by (3.6) satisfies the condition $F(x,0)=F(x,1)=0$. In this case, $U_t\widetilde{f\cdot 2}(x,0,0)$ exists for small t and $0\leqq f\in\bar{C}^*(\bar{R}_d)$, and we have

$$U_t^0\widetilde{f\cdot 2}(x,0,0)=\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{(y-x)^2}{2t}}f(y)dy, \tag{4.4}$$

$$\int_S K((x,0,0),ds,d(y,p,0))2^pF(y,u(t-s,y))=\int_0^t ds\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi s}}e^{-\frac{(y-x)^2}{2s}}F(y,u(t-s,y))dy. \tag{4.5}$$

If we compare (3.3) with (4.3) and note (4.4) and (4.5), then we see that $u(t,x)=U_t\widetilde{f\cdot 2}(x,0,0)$ is the solution of (4.1) if $k(x)$ is constant and $F(u)=\sum_{n=1}^{\infty}(q_n^+-q_n^-)u^n$. When $F(u)$ is not analytic in u, we approximate F by polynomials F_n in $[0,1]$, satisfying $F_n(0)=F_n(\xi_n)=0$, $0<\xi_n<1$. Let $Z_t^{(n)}$ be the process which corresponds to F_n in the sense of Theorem 3.1, then we can define a sequence of

5) Here the dimension d is one. Trivially, this assumption is not essential.

$u_n(t, x; f) = U_t^{(n)}\widetilde{f\cdot 2}(x, 0, 0)$, $U_t^{(n)}$ denoting the semi-group for $Z_t^{(n)}$, which is the solution of (4.1) replaced F by F_n. Using the fact that F_n converges to F uniformly in $[0, 1]$, we can see, for any $T>0$, $u_n(t, x; f)$ is an uniformly convergent sequence in $t \in [0, T]$. This shows that the solution of (4.1) with the initial value $0 \leqq f \in \bar{C}^*(\bar{R}_d)$ can be expressed as the limit of $U_t^{(n)}\widetilde{f\cdot 2}(x, 0, 0)$.

Remark. If we consider a diffusion process, instead of Brownian motion, whose generator is $A=\sum_{i,j} a_{ij}(x)\, \partial^2/\partial x_i \partial x_j$ with some conditions on a_{ij}, then we can treat the solution of

$$\frac{\partial u}{\partial t} = \sum_{ij} a_{ij}(x) \partial^2 u/\partial x_i \partial x_j + F(u).$$

5. Remark to construction of the processes. We shall give some comments about construction of the processes treated above. Starting from the given quantities, that is, Brownian motion, $k(x)$, and $(q_n^+(x), q_n^-(x))$, we can construct the process satisfying the condition 2 and (4.4). To do this, we can adopt the method of N. Ikeda, M. Nagasawa, and S. Watanabe [4]. The key points are to construct an integral kernel appropriately and to check some analytical conditions. Then Moyal's results [7] are ready to apply for it. The author has been communicated from M. Nagasawa that the processes discussed above can be constructed probabilistically in general set up.

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