189. On the Absolute Logarithmic Summability of the Allied Series of a Fourier Series

By Fu YEH
Department of Mathematics, Tsing Hua University, Taiwan, China

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1. **Introduction.** § 1.1. **Definition.***) *Let $\lambda=\lambda(w)$ be continuous, differentiable and monotone increasing in $(0, \infty)$, and let it tend to infinity as $w\rightarrow\infty$. For a given series $\sum_1^\infty a_n$, put*

$$C_r(w)=\sum_{n\leqslant w}\{\lambda(w)-\lambda(n)\}^r a_n \qquad (r\geqslant 0).$$

Then the series $\sum_1^\infty a_n$ is called to be summable $|R, \lambda, r|$ $(r\geqslant 0)$, if for a positive number A,

$$\int_A^\infty \left| d\left[\frac{C_r(w)}{\{\lambda(w)\}^r}\right]\right| <\infty.$$

For $r>0$, we have

$$\frac{d}{dw}\left[\frac{C_r(w)}{\{\lambda(w)\}^r}\right]=\frac{r\lambda'(w)}{\{\lambda(w)\}^{1+r}}\sum_{n\leqslant w}\{\lambda(w)-\lambda(n)\}^{r-1}\lambda(n)a_n.$$

Hence $\sum_1^\infty a_n$ is summable $|R, \lambda, r|$ $(r>0)$, if and only if

$$\int_A^\infty \left|\frac{r\lambda'(w)}{\{\lambda(w)\}^{1+r}}\sum_{n\leqslant w}\{\lambda(w)-\lambda(n)\}^{r-1}\lambda(n)a_n\right| dw<\infty.$$

§ 1.2. We suppose that $f(t)$ is integrable in the Lebesgue sense in the interval $(-\pi, \pi)$, and is periodic with period 2π, so that

$$f(t)\sim\tfrac{1}{2}a_0+\sum_1^\infty (a_n \cos nt+b_n \sin nt)=\tfrac{1}{2}a_0+\sum_1^\infty A_n(t).$$

Then the allied series is

$$\sum_1^\infty (b_n \cos nt-a_n \sin nt)=\sum_1^\infty B_n(t).$$

We write

(1) $$\psi(t)=\tfrac{1}{2}\{f(x+t)-f(x-t)\},$$

(2) $$\psi_1(t)=\frac{1}{\log(2\pi/t)}\int_t^\pi \frac{\psi(u)}{u}du.$$

In my thesis [2], I have proved that, if $t^{-1}\psi_1(t)\left(\log\frac{2\pi}{t}\right)^2$ is integrable in $(0, \pi)$, then the allied series of the Fourier series of $f(t)$ is summable $|R, \log w, 2|$. The object of the present paper is to prove the following

Theorem. *If the integral $\int_0^\pi t^{-1}|d\psi_1(t)|$ exists, then the allied*

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*) Mohanty [1].

series of the Fourier series of $f(t)$, at $t=x$, is summable $|R, \log w, 1+\delta|$, *where* $0<\delta<1$.

2. Proof of the theorem. **§ 2.1.** We write

$$g(w, t)=\sum_{n\leqslant w}\log n\left(\log\frac{w}{n}\right)^{\delta}\sin nt, \tag{3}$$

$$h(w, t)=\sum_{n\leqslant w}n^{-1}\log n\left(\log\frac{w}{n}\right)^{\delta}\sin^2\frac{nt}{2}. \tag{4}$$

For the proof of the theorem we require the following lemmas:

Lemma 1. $g(w, t)=O(w\log w)$.

Proof. By (3), we write

$$|g(w, t)|\leqslant\sum_{n\leqslant w}\log n\left(\log\frac{w}{n}\right)^{\delta}=\sum_{n<w^{1/(1+\delta)}}+\sum_{w^{1/(1+\delta)}\leqslant n\leqslant w}=P+Q.$$

By the second mean value theorem, we have

$$P\leqslant\int_1^{w^{1/(1+\delta)}}\log u\left(\log\frac{w}{u}\right)^{\delta}du+O(\log w(\log w)^{\delta})$$
$$\leqslant\log w(\log w)^{\delta}w^{1/(1+\delta)}+O((\log w)^{1+\delta})=O(w^{1/(1+\delta)}(\log w)^{1+\delta}),$$
$$Q\leqslant\int_{w^{1/(1+\delta)}}^{w}\log u\left(\log\frac{w}{u}\right)^{\delta}du+O((\log w)^{1+\delta})$$
$$<\log w\int_{w^{1/(1+\delta)}}^{w}\left(\log\frac{w}{u}\right)^{\delta}du+O((\log w)^{1+\delta})=O(w\log w).$$

Thus it follows that $g(w, t)=O(w\log w)$.

Lemma 2. $g(w, t)=O(t^{-1}(\log w)^{1+\delta})$.

Proof. By Abel's lemma, we have

$$g(w, t)=\sum_{n\leqslant w}\log n\left(\log\frac{w}{n}\right)^{\delta}\sin nt$$
$$\leqslant\frac{A}{t}\sum_{n\leqslant w-1}\left|\varDelta\left(\log n\left(\log\frac{w}{n}\right)^{\delta}\right)\right|+O(w^{-\delta}t^{-1}\log w)$$
$$\leqslant\frac{A}{t}\left\{\sum_{n<w^{1/(1+\delta)}}+\sum_{w^{1/(+\delta)}\leqslant n\leqslant w-1}\right\}+O(t^{-1}w^{-\delta}\log w)=O(t^{-1}(\log w)^{1+\delta}).$$

Lemma 3. $g(w, t)=O(t^{-2}(\log w)^{\delta})$.

Proof. By twice use of Abel's lemma, we get

$$g(w, t)=\sum_{n\leqslant w}\log n\left(\log\frac{w}{n}\right)^{\delta}\sin nt$$
$$=\sum_{n\leqslant w-2}\varDelta^2\left(\log n\left(\log\frac{w}{n}\right)^{\delta}\right)\sum_1^n\tilde{D}_v(t)+O(t^{-2}w^{-\delta}\log w)+O(t^{-1}w^{-\delta}\log w),$$

where $\tilde{D}_v(t)$ is the v-th conjugate Dirichlet kernel, and hence

$$|g(w, t)|\leqslant\frac{A}{t^2}\sum_{n\leqslant w-1}\left|\varDelta^2\left(\log n\left(\log\frac{w}{u}\right)^{\delta}\right)\right|+O(t^{-2}w^{-\delta}\log w) \tag{5}$$
$$=\frac{A}{t^2}\left\{\sum_{n<ew^{1/(1+\delta)}}+\sum_{ew^{1/(1+\delta)}\leqslant n<e^{\delta-1}w}+\sum_{e^{\delta-1}w\leqslant n\leqslant w-2}\right\}+O(t^{-2}w^{-\delta}\log w)$$
$$=\frac{A}{t^2}(P+Q+R)+O(t^{-2}w^{-\delta}\log w).$$

We have

$$(6)\qquad P=\sum_{n<ew^{1/(1+\delta)}}\left|\Delta^2\left(\log n\left(\log\frac{w}{n}\right)^{\delta}\right)\right|=O((\log w)^{\delta}),$$

$$(7)\qquad Q=\sum_{ew^{1/(1+\delta)}\leqslant n<e^{\delta-1}w}\left|\Delta^2\left(\log n\left(\log\frac{w}{n}\right)^{\delta}\right)\right|=O(w^{-1/(1+\delta)}(\log w)^{\delta}),$$

$$(8)\qquad R=\sum_{e^{\delta-1}w\leqslant n\leqslant w-2}\left|\Delta^2\left(\log n\left(\log\frac{w}{n}\right)^{\delta}\right)\right|=O(w^{-1}\log w).$$

Hence the desired result follows from (5), (6), (7), and (8).

Lemma 4. $h(w, t)=O((\log w)^{2+\delta})$.

Proof. From (4) we get

$$|h(w,t)|\leqslant\sum_{n\leqslant w}n^{-1}\log n\left(\log\frac{w}{n}\right)^{\delta}\leqslant\int_1^w x^{-1}\log x\left(\log\frac{w}{x}\right)^{\delta}dx+O(w^{-2}\log w)$$

$$<\log w\int_1^w\left(\log\frac{w}{x}\right)^{\delta}x^{-1}dx+0(w^{-2}\log w)=O((\log w)^{2+\delta}).$$

Lemma 5. $h(w, t)=O(t^{-2}(\log w)^{\delta})$.

Proof. By Abel's lemma, we have

$$h(w,t)=\sum_{n\leqslant w}\frac{\log n}{n}\left(\log\frac{w}{n}\right)^{\delta}\sin^2\frac{nt}{2}$$

$$=\sum_{n\leqslant w-1}\Delta\left(\frac{\log n}{n}\left(\log\frac{w}{n}\right)^{\delta}\right)\sum_1^n\sin^2\frac{vt}{2}$$

$$+\Delta\left(\frac{\log[w]}{[w]}\left(\log\frac{w}{[w]}\right)^{\delta}\right)\sum_1^{[w]}\sin^2\frac{vt}{2}.$$

Since

$$\sum_{v=1}^n\sin^2\frac{vt}{2}=\sum_{v=1}^n\sin\frac{t}{2}\sum_{\mu=1}^{v}\sin(\mu+\tfrac{1}{2})t,$$

we have

$$\left|\sum_{v=1}^n\sin^2\frac{vt}{2}\right|\leqslant\frac{A}{t^2}.$$

Thus

$$|h(w,t)|\leqslant\frac{A}{t^2}\sum_{n\leqslant w-1}\left|\Delta\left(n^{-1}\log n\left(\log\frac{w}{n}\right)^{\delta}\right)\right|+O(t^{-2}w^{-2-\delta}\log w).$$

We have

$$\sum_{n\leqslant w-1}\left|\Delta\left(n^{-1}\log n\left(\log\frac{w}{n}\right)^{\delta}\right)\right|=\sum_{n<e}+\sum_{e\leqslant n\leqslant w-1}=O((\log w)^{\delta})+O(w^{-2-\delta}\log w).$$

Hence the desired result follows.

§ 2.2. We shall now prove the theorem. By integrating by parts twice, we get

$$(9)\qquad B_n(x)=\frac{2}{\pi}\int_0^{\pi}\psi(t)\sin nt\,dt$$

$$=\frac{2}{\pi}\int_0^{\pi}\psi_1(t)\log\frac{2\pi}{t}(t\sin nt)'dt$$

$$=-\frac{2}{\pi}\int_0^\pi d\psi_1(t)\Big[t\log\frac{2\pi}{t}\sin nt-n^{-1}\cos nt+n^{-1}\Big]$$

$$=-\frac{2}{\pi}\int_0^\pi d\psi_1(t)\Big[t\log\frac{2\pi}{t}\sin nt+2n^{-1}\sin^2\frac{nt}{2}\Big]=u_n+v_n.$$

The series $\sum_1^n u_n$ is summable $|R, \log w, 1+\delta|$ if

$$I_1=\int_e^\infty\frac{(1+\delta)dw}{w(\log w)^{2+\delta}}\Big|\sum_{n\leqslant w}\log n\Big(\log\frac{w}{n}\Big)^\delta u_n\Big|<\infty.$$

Substituting for u_n from (9), we have, by (8)

$$I_1\leqslant\frac{2(1+\delta)}{\pi}\int_0^\pi|d\psi_1(t)|\,t\log\frac{2\pi}{t}\int_e^\infty\frac{dw}{w(\log w)^{2+\delta}}|g(w,t)|. \tag{10}$$

Since $\int_0^\pi t^{-1}|d\psi_1(t)|$ is finite, it is sufficient to show that

$$J_1=\int_e^\infty\frac{dw}{w(\log w)^{2+\delta}}|g(w,t)|=O\Big(1/t^2\log\frac{2\pi}{t}\Big)\quad\text{for}\quad 0<t<\pi.$$

Let

$$J_1=\int_e^\infty=\int_e^{\frac{2\pi}{t}\left(\log\frac{2\pi}{t}\right)}+\int_{\frac{2\pi}{t}\left(\log\frac{2\pi}{t}\right)}^{e^{2\pi/t}}+\int_{e^{2\pi/t}}^\infty=J_{11}+J_{12}+J_{13}.$$

By Lemma 1, we have

$$J_{11}=O\Big(\int_e^{\frac{2\pi}{t}\left(\log\frac{2\pi}{t}\right)}\frac{dw}{(\log w)^{1+\delta}}\Big)=O\Big(t^{-1}\Big(\log\frac{2\pi}{t}\Big)^{-1}\Big).$$

By Lemma 2, we have

$$J_{12}=O\Big(t^{-1}\int_{\frac{2\pi}{t}\left(\log\frac{2\pi}{t}\right)}^{e^{2\pi/t}}\frac{dw}{w\log w}\Big)=O\Big(t^{-1}\log\frac{2\pi}{t}\Big).$$

By Lemma 3, we have

$$J_{13}=O\Big(t^{-2}\int_{e^{2\pi/t}}^\infty\frac{dw}{w(\log w)^2}\Big)=O(t^{-1}).$$

Hence we get $J_1=O\Big(t^{-1}\log\frac{2\pi}{t}\Big)=O\Big(t^{-2}/\log\frac{2\pi}{t}\Big)$.

It remains to prove that the series $\sum_1^\infty v_n$ is summable $|R, \log w, 1+\delta|$. The series $\sum_1^\infty v_n$ is summable $|R, \log w, 1+\delta|$ if

$$I_2=\int_e^\infty\frac{(1+\delta)dw}{w(\log w)^{2+\delta}}\Big|\sum_{n\leqslant w}\log n\Big(\log\frac{w}{n}\Big)^\delta v_n\Big|<\infty.$$

Subtituting for v_n from (9), we have, by (4),

$$I_2\leqslant\frac{2(1+\delta)}{\pi}\int_0^\pi|d\psi_1(t)|\int_e^\infty\frac{dw}{w(\log w)^{2+\delta}}|h(w,t)|.$$

Since $\int_0^\pi t^{-1}|d\psi_1(t)|$ is finite, it is enough to show that

$$J_2=\int_e^\infty\frac{dw}{w(\log w)^{2+\delta}}|h(w,t)|=O(t^{-1}).$$

Let

$$J_2=\int_e^\infty=\int_e^{e^{2\pi/t}}+\int_{e^{2\pi/t}}^\infty=J_{21}+J_{22}.$$

By Lemma 4, we have

$$J_{21}=O\left(\int_e^{e^{2\pi/t}}\frac{dw}{w}\right)=O(t^{-1}).$$

By Lemma 5, we have

$$J_{22}=O\left(t^{-2}\int_{e^{2\pi/t}}^\infty\frac{dw}{w(\log w)^2}\right)=O(t^{-1}).$$

Hence we get $J_2=O(t^{-1})$. Thus the proof of the theorem is completed.

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References

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