223. Some Notes on the Cluster Sets of Meromorphic Functions

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1. Let D be a domain in the z-plane, Γ its boundary, E a totally disconnected compact set on Γ and z_0 a point of E such that $U(z_0)\cap(\Gamma-E)\neq\emptyset$ for any neighborhood $U(z_0)$ of z_0. We consider a normal exhaustion $\{F_n\}$ of the complementary domain F of E with respect to the extended z-plane and the graph $0<u<R$, $0<v<2\pi$ associated with this exhaustion in Noshiro's sense [3], where R is the length of this graph and may be infinite. The niveau curve $u(z)=r(0<r<R)$ on F consists of a finite number of closed analytic curves $\gamma_r^{(i)}(i=1, 2, \cdots, m(r))$ and we set

$$\Lambda(r)=\max_{1\leq i\leq m(r)}\int_{\gamma_r^{(i)}}dv.$$

Now suppose that there exists an exhaustion $\{F_n\}$ with the graph satisfying

$$(1)\qquad \limsup_{r\to R}\,(R-r)\int_0^r\frac{dr}{\Lambda(r)}=\infty.$$

Then the integral $\int_0^R \exp\left(2\pi\int_0^r\frac{dr}{\Lambda(r)}\right)dr$ diverges, so that the complementary domain F of E belongs to the class O_{AB}^0 (see Kuroda [1]), i.e., E belongs to the class $N_{\mathfrak{B}}^0$ in the sense of Noshiro [4]. Therefore, for any single-valued meromorphic function $w=f(z)$ in D, the set $\Omega=C_D(f, z_0)-C_{\Gamma-E}(f, z_0)$ is empty or open and each value α belonging to $\Omega-R_D(f, z_0)$ is an asymptotic value of $w=f(z)$ at z_0 or there is a sequence of points $\zeta_n\in E$ tending to z_0 such that α is an asymptotic value of $f(z)$ at each ζ_n. Further $\Omega-R_D(f, z_0)$ is an at most countable union of sets of the class $N_{\mathfrak{B}}$. (These three facts have been proved by Noshiro in his recent paper [4].) We shall restrict our consideration to the case where E is contained in a single boundary component Γ_0 of Γ. Then we have

Theorem 1. *Suppose that Ω is not empty. If E is contained in a single boundary component Γ_0 of Γ and there exists an exhaustion $\{F_n\}$ with the graph satisfying (1), then $w=f(z)$ takes on every value, with two possible exceptions, belonging to any component Ω_n of Ω, infinitely often in the intersection of any neighborhood of z_0 and D.*

In the special case where D is simply connected, we have

Theorem 2. *Suppose that D is simply connected and $w=f(z)$ is regular in the intersection of some neighborhood of z_0 and D. Then, under the same assumptions as in Theorem 1, $w=f(z)$ takes on every finite value, with one possible exception, belonging to any component Ω_n of Ω infinitely often in the intersection of any neighborhood of z_0 and D.*

Remark 1. If E is of logarithmic capacity zero, then there exists an exhaustion $\{F_n\}$ with the graph, its length being infinite. Hence the condition (1) is satisfied and we see that Theorems 1 and 2 are extensions of Noshiro's theorem [2].

2. We assume that E contains at least two points. Without any loss of generality, we may assume that an exceptional value w_0 in Ω_n, if exists, is finite. Inside Ω_n we draw a simple closed analytic curve C which does not pass through the point at infinity and encloses w_0 and whose interior consists of only interior points of Ω_n. We select a positive number η less than the diameter of Γ_0 such that $f(z)\neq w_0$ in the common part of D and (K): $|z-z_0|<\eta$ and the closure M_η of the union $\bigcup C_D(f,\zeta)$ for all ζ belonging to the intersection of $\Gamma-E$ with $(\overline{K})$ lies outside C. We draw in (K) a simple closed analytic curve γ which encloses z_0 and does not pass through any point of E. Since w_0 is either an asymptotic value of $w=f(z)$ at z_0 or there exists a sequence $z'_n\in E$ tending to z_0 such that w_0 is an asymptotic value at each z'_n, it is possible to find a point z'_0 (may be z_0) belonging to $E\cap(\gamma)$, (γ) being the interior of γ, such that w_0 is an asymptotic value of $w=f(z)$ at z'_0. Let Λ be the asymptotic path with the asymptotic value w_0 at z'_0. We may assume that the image of Λ under $w=f(z)$ is a curve lying completely inside C. Considering the open set of points z in the intersection of D and (γ) such that $w=f(z)$ lies inside C, we denote by Δ its component containing the path Λ. As is easily seen, the boundary of Δ consists of a finite number of arcs on γ, at most a countable number of analytic curves (relative boundary) inside $D\cap(\gamma)$, and a closed subset E_0 of E.

Let r_0 be a fixed positive number such that for $r_0\leq r<R$ all the level curve γ_r does not intersect γ and does the asymptotic path Λ. We take the component γ_r^0 of γ_r (one of $\gamma_r^{(i)}(i=1,2,\cdots,m(r))$ enclosing z'_0 and denote Θ_r^0 the common part of γ_r^0 and the domain Δ; Θ_r^0 consists of only a finite number of cross-cuts because we have taken η less than the diameter of Γ_0. Denote by $\Delta(r)$ the common part of Δ and the exterior of γ_r, by $A(r)$ the area of the Riemannian image of the open set $\Delta(r)$ under the function $w=f(z)$ and by $L^0(r)$ the total length of the image of Θ_r^0. Then, using the local parameter

$\zeta=u+iv$, we have

$$L^0(r)=\int_{\Theta_r^0}|f'|\,dv.$$

Denote by $\delta>0$ the distance of C from the image of Δ. Then, a geometric consideration gives $L^0(r)\geqq 2\delta$ for $r_0\leq r<R$ and by Schwarzś inequality, we have

$$4\delta^2\leqq L^0(r)^2\leqq \Delta(r)\int_{\Theta_r^0}|f'|^2dv\leqq \Delta(r)\int_{u=r}|f'|^2dv. \tag{2}$$

Note that

$$\int_{u=r}|f'|^2dv=\frac{dA(r)}{dr}. \tag{3}$$

From (2) and (3) we have

$$4\delta^2\int_0^r\frac{dr}{\Delta(r)}\leq A(r)-A(0), \tag{4}$$

so that our condition (1) gives

$$\lim_{r\to R}A(r)=\infty. \tag{5}$$

Next we shall prove that the regularly exhaustibility condition in Ahlfors' sense is satisfied. Denoting by $L(r)$ the total length of the image of Θ_r, the common part of γ_r and Δ, we have

$$L(r)^2\leqq 2\pi\frac{dA(r)}{dr}. \tag{6}$$

Now, contrary, suppose that

$$\liminf_{r\to R}L(r)/A(r)\geq\sigma>0. \tag{7}$$

Then from (6) and (7) we see

$$\frac{\sigma^2}{2\pi}(R-r)=\frac{\sigma^2}{2\pi}\int_r^R dr\leq\int_r^R\frac{dA(r)}{A(r)^2}=\frac{1}{A(r)}, \tag{8}$$

since $A(R)=\infty$ by (5). Using (4), we have thus

$$\frac{2\sigma^2\delta^2}{\pi}(R-r)\int_0^r\frac{dr}{\Delta(r)}\leqq 1. \tag{9}$$

This contradicts our condition (1) and the regularly exhaustibility condition must hold.

3. Now it is easy to prove our theorem. Indeed, we need only to follow Noshiro's arguments [2]. For completeness, we shall give proofs in the below. Because of Noshiro's theorem, it is enough for us to prove the theorem under the condition that E contains at least two points.

Proof of Theorem 1. Contrary to our assertion, we suppose that there are three exceptional values w_0, w_1, and w_2 in Ω_n, where it does not bring any loss of generality if we assume these three values are finite. Inside Ω_n we draw a simple closed analytic curve C which encloses w_0, w_1 and passes through w_2 but not through the point at infinity and whose interior consists of only interior points of Ω_n. We select a positive number η less than the diameter of Γ_0 such that $f(z) \neq w_0, w_1$, and w_2 in the common part of D and (K): $|z-z_0|<\eta$ and the closure M_η lies outside C. We determine γ, Λ, and Δ by the same way as in § 2 and for them we take r_0. We shall show that Δ is simply-connected. Note that the boundary of Δ does not contain any closed analytic curve, since any analytic curve in the boundary of Δ is transformed by $w=f(z)$ into a curve lying on the simple closed curve C passing through the exceptional value w_2. Further, the boundary of the bounded domain Δ consists of a single continuum, since E is contained in a single component Γ_0 of Γ. Thus it is concluded that Δ is simply connected. Now it is clear that the open set $\Delta(r)$, $r_0 \leq r < R$, consists of simply connected components, because Θ_r does not contain any loop-cut. We denote these components by $\Delta^{(i)}(r)$ $(i=1, 2, \cdots, p(r))$. Denote by $\Phi^{(i)}(r)$ the Riemannian image of $\Delta^{(i)}(r)$ under $w=f(z)$ $(i=1, 2, \cdots, p(r))$. If we denote by Φ_0 the domain obtained by excluding the two points w_0 and w_1 from the interior of C, then, by hypothesis, $\Phi^{(i)}(r)$ is a finite covering surface of the base surface Φ_0 $(i=1, 2, \cdots, p(r))$. By Ahlfors' principal theorem on covering surfaces, we have

$$S^{(i)}(r) \leqq hL^{(i)}(r) \qquad (i=1, 2, \cdots, p(r)), \tag{10}$$

where $S^{(i)}(r)$ denotes the average number of sheets of $\Phi^{(i)}(r)$, i.e., $S^{(i)}(r)$ denotes the ratio between the area of $\Phi^{(i)}(r)$ and the area of Φ_0, $L^{(i)}(r)$ the length of the boundary of $\Phi^{(i)}(r)$ relative to Φ_0, and h is a constant dependent only upon Φ_0. From (10)

$$\sum_{i=1}^{p(r)} S^{(i)}(r) \leqq h \sum_{i=1}^{p(r)} L^{(i)}(r),$$

that is,

$$S(r) \leqq h(L(r)+L_0), \tag{11}$$

where L_0 denotes the total length of the image of the arcs of γ included in the boundary of Δ. Accordingly

$$\liminf_{r \to R} \frac{L(r)}{S(r)} \geqq \frac{1}{h} > 0, \tag{12}$$

while we have showed in § 2 that the regularly exhaustibility condition holds. Contradiction. Our theorem must be true.

Proof of Theorem 2. Suppose that there are two finite exceptional values w_0 and w_1 within Ω_n, and let C be any simple closed analytic curve in Ω_n, which surrounds w_0 and w_1 and whose interior consists of only interior points of Ω_n. Let Δ be the domain defined in the same way as in the proof of Theorem 1. Then, we can easily see that Δ is also simply connected, for if Δ were not simply connected, the boundary of Δ would contain at least one closed analytic contour q such that q is a loop-cut of D. Hence $w=f(z)$ would take inside q a value lying outside the simple closed curve

C, while $w=f(z)$ is regular both inside and on q, and the image of q by $w=f(z)$ would lie on C. This is a contradition. Repeating the same argument as in the proof of Theorem 1, we complete the proof.

4. In this section we shall give an example of E satisfying the condition (1) by means of Cantor sets. We prove.

Theorem 3. *Let E be a Cantor set on the interval I_0: $[-1/2, 1/2]$ on the real axis of the z-plane with successive ratios ξ_n, $0<\xi_n=2l_n<2/3$. If*

$$\limsup_{n\to\infty}\Big(\sum_{p=n+1}^{\infty}\frac{\log\xi_p^{-1}}{2^p}\Big)\Big(\sum_{p=1}^{n}\log\xi_p^{-1}\Big)=\infty, \tag{13}$$

then there exists an exhaustion $\{F_n\}$ of the complementary domain F of E, the graph associated with which satisfies the condition (1).

Proof. Defining the Cantor set E, we repeat successively to exclude an open segment from the center of another segment and there remain 2^n segments of equal length $\prod_{p=1}^{n}l_p$ after we repeat n times, beginning with the interval I_0. We denote by $I_{n,k}$ $(n=1, 2, \cdots; k=1, 2, \cdots, 2^n)$ these segments and by $C_{n,k}$ $(n=1, 2, \cdots; k=1, 2, \cdots, 2^n)$ the circles $|z-z_{n,k}|=(\prod_{p=1}^{n-1}l_p)(1-l_n)/2$, where $z_{n,k}$ are the middle points of $I_{n,k}$. Supposing that $C_{n,k}$ encloses $C_{n+1,2k-1}$ and $C_{n+1,k}$, we see that these two circles touch outside each other, and denote by $S_{n,k}$ $(n=1, 2, \cdots; k=1, 2, \cdots, 2^n)$ the ring domains bounded by $C_{n,k}$ and $C_{n+1,k-1}\cup C_{n+1,k}$. The harmonic modulus μ_n of $S_{n,k}$ is greater than $\log(2\xi_n^{-1}/3)$. We define an exhaustion $\{F_n\}$ of F as follows. The outside of the circle $|z|=2$ is taken as F_0 and the common part of the outsides of all the $C_{n,k}$ $(k=1, 2, \cdots, 2^n)$ is taken as F_n. Then, for each n, the open set $F_{n+1}-\bar{F}_n$ consists of ring domains $S_{n,k}$ $(k=1, 2, \cdots, 2^n)$, so that its harmonic modulus σ_n is equal to $\mu_n/2^n$. Hence the length R of the graph associated with this $\{F_n\}$ is

$$\sum_{p=0}^{\infty}\sigma_p=\sum_{p=0}^{\infty}\frac{\mu_p}{2^p},$$

where $\sigma_0=\mu_0$ is the harmonic modulus of the ring domain $F_1-\bar{F}_0$. It is easily seen that

$$\Lambda(r)=\frac{2\pi}{2^n}\quad\text{if}\quad\sum_{p=0}^{n}\sigma_p<r\leqq\sum_{p=0}^{n+1}\sigma_p.$$

Hence, if $r=\sum_{p=0}^{n}\sigma_p$, then

$$R-r=\sum_{p=n+1}^{\infty}\frac{\mu_p}{2^p}\geqq\sum_{p=n+1}^{\infty}\frac{\log\xi_p^{-1}}{2^p}+\frac{\log(2/3)}{2^n} \tag{14}$$

and

$$\int_0^r\frac{dr}{\Lambda(r)}=\frac{1}{2\pi}\sum_{p=0}^{n}2^p\sigma_p=\frac{1}{2\pi}\sum_{p=0}^{n}\mu_p>\frac{1}{2\pi}\sum_{p=1}^{n}\log\xi_p^{-1}+\frac{n}{2\pi}\log(2/3). \tag{15}$$

Therefore it is enough for us to show that the condition (1) holds when $R<\infty$. Then $(\log(2/3))^2 n/2^{n+1}\pi \to 0$ as $n\to\infty$ and

$$\left|(\log(2/3))(1/2^{n+1}\pi)\sum_{p=1}^{n}\log\xi_p^{-1}\right| \leqq |\log(2/3)|\, R/2\pi = O(1).$$

Further we see from (13) that $n\Big/\sum_{p=1}^{n}\log\xi_p^{-1}\to 0$ as $n\to\infty$. Hence, from (14) and (15),

$$(R-r)\int_0^r\frac{dr}{\Lambda(r)}\geqq\Big(\sum_{p=n+1}^{\infty}\frac{\log\xi_p^{-1}}{2^p}\Big)\Big(\sum_{p=1}^{n}\log\xi_p^{-1}\Big)\Big(\frac{1}{2\pi}(1-o(1))\Big)+O(1),$$

so that, by making $n\to\infty$, we see that the condition (1) holds.

Example. If successive ratios ξ_n satisfy

(16) $\xi_{n+1}=O(\xi_n^{\lambda})$ with $\lambda>\sqrt{2}$ and $n=1, 2, \cdots$,

then they satisfy (13).

Remark 2. It is well-known that a Cantor set E is of logarithmic capacity zero if and only if

$$\sum_{p=1}^{\infty}\frac{\log\xi_p^{-1}}{2^p}=\infty.$$

Hence we see that there exist ones of positive logarithmic capacity among Cantor sets satisfying (16) for λ, $2>\lambda>\sqrt{2}$.

References

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