219. A Characterization of Axiom Schema Playing the Rôle of Tertium non Datur in Intuitionistic Logic

By Masazumi HANAZAWA

Department of Mathematics, Tokyo University of Education

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As is well-known, there are some axiom schemas, by each of which a system of classical logic is obtained from any system of intuitionistic logic.

$A \vee \neg A$ (tertium non datur),

$\neg\neg A \to A$ (discharge of double negation)

and

$((A \to B) \to A) \to A$ (Peirce's law)

are famous examples among them. The purpose of this paper is to give a criterion for those axiom schemas, in the scope of *propositional logic.*

Main result. Let us consider the three-valued logic defined by the following truth-tables:

$A \wedge B$

A \ B	t	u	f
t	t	u	f
u	u	u	f
f	f	f	f

$A \vee B$

A \ B	t	u	f
t	t	t	t
u	t	u	u
f	t	u	f

$A \to B$

A \ B	t	u	f
t	t	u	f
u	t	t	f
f	t	t	t

$\neg A$

A	
t	f
u	f
f	t

where truth-values t, f, and u correspond to 'true', 'false', and 'unknown',[1] respectively. Then our main result can be stated as follows:

If and only if a formula $\mathfrak{A}$ *is a tautology in usual sense (or*

1) The truth-value u is not exactly corresponding to the usual meaning of the word "unknown".

in the usual two-valued logic) and is not identically true in the above-mentioned three-valued logic, the classical propositional calculus is obtained from the intuitionistic propositional calculus by adjoining $\mathfrak{A}$ *as an axiom schema.*

In the following, by "t-formula" we shall mean such a formula as is identicaly true in that three-valued logic. Then our main result is divided into the following two theorems:

Theorem 1. *If the classical propositional calculus is obtained from the intiutionistic propositional calculus by adjoining* $\mathfrak{A}$ *as an axiom schema, then* $\mathfrak{A}$ *is not a* t*-formula.*

Theorem 2. *If* $\mathfrak{A}$ *is a tautology and is not a* t*-formula, then the classical propositional calculus is obtained from the intuitionistic propositional calculus by adjoining* $\mathfrak{A}$ *as an axiom schema.*

1. Proof of Theorem 1. Every axiom of the intuitionistic propositional calculus is a t-formula, and also the result obtained from a t-formula by substituting arbitrary formulas for propositional variables is a t-formula. By every rule of inference in the intuitionistic propositional calculus (e.g. modus ponens), from one or more t-formulas we infer a t-formula. Then the provable formulas in the system obtained from the intuitionistic propositional calculus by adjoining some t-formulas as axiom schemas are all t-formulas.

On the other hand, there is such a tautology as is not a t-formula. For example, the tautology

$$\neg\neg A \to A$$

is not a t-formula.

Let the system S obtained from the intuitionistic propositional calculus by adjoining a formula $\mathfrak{A}$ as an axiom schema be equivalent to classical. Then every tautology is provable in S. If $\mathfrak{A}$ were a t-formula, then the tautology

$$\neg\neg A \to A$$

which is provable in S would be a t-formula. Hence $\mathfrak{A}$ can not be a t-formula, q.e.d.

2. Proof of Theorem 2. 2.1. Lemma 1. *Let* $\mathfrak{A}$ *be a formula containing only one propositional variable* A*. Then one of the formulas*

$$\neg\neg A \to \mathfrak{A}, \qquad \neg\neg A \to \neg\mathfrak{A}$$

and

$$\neg\neg A \to (\mathfrak{A} \leftrightarrow A)$$

is provable intuitionistically.

This lemma is easily proved by mathematical induction on the number of logical symbols contained in $\mathfrak{A}$, and by help of the intuitionistic provability of formulas of the following forms:

$$\mathfrak{B}\rightarrow[\mathfrak{C}\rightarrow(\mathfrak{B}\wedge\mathfrak{C})]$$
$$\neg\mathfrak{B}\rightarrow\neg(\mathfrak{B}\wedge\mathfrak{C}),$$
$$\neg\mathfrak{C}\rightarrow\neg(\mathfrak{B}\wedge\mathfrak{C}),$$
$$\mathfrak{B}\rightarrow\{(\mathfrak{C}\leftrightarrow A)\rightarrow[(\mathfrak{B}\wedge\mathfrak{C})\leftrightarrow A]\},$$
$$(\mathfrak{B}\leftrightarrow A)\rightarrow\{\mathfrak{C}\rightarrow[(\mathfrak{B}\wedge\mathfrak{C})\leftrightarrow A]\},$$
$$(\mathfrak{B}\leftrightarrow A)\rightarrow\{(\mathfrak{C}\leftrightarrow A)\rightarrow[(\mathfrak{B}\wedge\mathfrak{C})\leftrightarrow A]\},$$
$$\mathfrak{B}\rightarrow(\mathfrak{B}\vee\mathfrak{C}),$$
$$\mathfrak{C}\rightarrow(\mathfrak{B}\vee\mathfrak{C}),$$
$$\neg\mathfrak{B}\rightarrow[\neg\mathfrak{C}\rightarrow\neg(\mathfrak{B}\vee\mathfrak{C})],$$
$$\neg\mathfrak{B}\rightarrow\{(\mathfrak{C}\leftrightarrow A)\rightarrow[(\mathfrak{B}\vee\mathfrak{C})\leftrightarrow A]\},$$
$$(\mathfrak{B}\leftrightarrow A)\rightarrow\{\neg\mathfrak{C}\rightarrow[(\mathfrak{B}\vee\mathfrak{C})\leftrightarrow A]\},$$
$$(\mathfrak{B}\leftrightarrow A)\rightarrow\{(\mathfrak{C}\leftrightarrow A)\rightarrow[(\mathfrak{B}\vee\mathfrak{C})\leftrightarrow A]\},$$
$$\mathfrak{C}\rightarrow(\mathfrak{B}\rightarrow\mathfrak{C}),$$
$$\neg\mathfrak{B}\rightarrow(\mathfrak{B}\rightarrow\mathfrak{C}),$$
$$\mathfrak{B}\rightarrow[\neg\mathfrak{C}\rightarrow\neg(\mathfrak{B}\rightarrow\mathfrak{C})],$$
$$\mathfrak{B}\rightarrow\{(\mathfrak{C}\leftrightarrow A)\rightarrow[(\mathfrak{B}\rightarrow\mathfrak{C})\leftrightarrow A]\},$$
$$\neg\neg A\rightarrow\{(\mathfrak{B}\leftrightarrow A)\rightarrow[\neg\mathfrak{C}\rightarrow\neg(\mathfrak{B}\rightarrow\mathfrak{C})]\},$$
$$(\mathfrak{B}\leftrightarrow A)\rightarrow[(\mathfrak{C}\leftrightarrow A)\rightarrow(\mathfrak{B}\rightarrow\mathfrak{C})],$$
$$\mathfrak{B}\rightarrow\neg\neg\mathfrak{B},$$
$$\neg\mathfrak{B}\rightarrow\neg\mathfrak{B},$$
$$\neg\neg A\rightarrow[(\mathfrak{B}\leftrightarrow A)\rightarrow\neg\neg\mathfrak{B}].$$

2.11. Corollary 1. *Let $\mathfrak{A}$ be a formula containing only one propositional variable A. If $\mathfrak{A}$ is a tautology and is not a* t*-formula, then*

$$\neg\neg A\rightarrow(\mathfrak{A}\leftrightarrow A)$$

is provable intuitionistically.

Proof. From the fact that $\mathfrak{A}$ is a tautology,

$$\neg\neg A\rightarrow\neg\mathfrak{A}$$

is not a t-formula, accordingly it is not provable intuitionistically. If

$$\neg\neg A\rightarrow\mathfrak{A}$$

were provable intuitionistically, then it would be a t-formula, and then $\mathfrak{A}$ would be a t-formula, because $\mathfrak{A}$ is a tautology. Hence

$$\neg\neg A\rightarrow(\mathfrak{A}\leftrightarrow A)$$

must be provable intuitionistically, q.e.d.

2.12. Corollary 2. *Let $\mathfrak{A}$ be a tautology containing only one propositional variable A and be not a* t*-formula. Then the system S obtained from the intuitionistic propositional calculus by adjoining $\mathfrak{A}$ as an axiom schema is equivalent to the classical propositional calculus.*

Proof. Firstly, let us remark the fact that the system S is a subsystem of the classical propositional calculus. Then we shall prove only the fact that classically provable formulas are all provable in S.

By Corollary 1, the formula

$$\neg\neg A \rightarrow (\mathfrak{A} \rightarrow A)$$

is provable intuitionistically, then so is

$$\mathfrak{A} \rightarrow (\neg\neg A \rightarrow A).$$

Accordingly, the *discharge of double negation*

$$\neg\neg A \rightarrow A$$

is provable in S, hence we can see the fact that all tautologies are provable in S, q.e.d.

2.2. **Lemma 2.** *Let $\mathfrak{A}(X_1, \cdots, X_n)$ be a tautology containing no propositional variable except $X_1, \cdots, X_n$, and be not a* $\mathfrak{t}$*-formula. Then there are appropriate formulas $\mathfrak{B}_1(A), \cdots, \mathfrak{B}_n(A)$, which contain no propositional variable except A, and*

$$\mathfrak{A}(\mathfrak{B}_1(A), \cdots, \mathfrak{B}_n(A))$$

is such a tautology as is not a $\mathfrak{t}$*-formula.*

Proof. It is clear that $\mathfrak{A}(\mathfrak{B}_1(A), \cdots, \mathfrak{B}_n(A))$ is a tautology. Then we shall prove only the fact that it is not a $\mathfrak{t}$-formula.

From the fact that $\mathfrak{A}(X_1, \cdots, X_n)$ is not a $\mathfrak{t}$-formula, there is such a valuation[2] $\mathfrak{v}$ as makes $\mathfrak{A}(X_1, \cdots, X_n)$ have a truth-value distinct from $\mathfrak{t}$. By $\mathfrak{v}(X)$ we mean the truth-value of X in $\mathfrak{v}$. Then we have

$$\mathfrak{A}(\mathfrak{v}(X_1), \cdots, \mathfrak{v}(X_n)) \neq \mathfrak{t}.$$

We define $\mathfrak{B}_i(A)$ by

$$\mathfrak{B}_i(A) = \begin{cases} \neg\neg A & \text{if } \mathfrak{v}(X_i) = \mathfrak{t}, \\ A & \text{if } \mathfrak{v}(X_i) = \mathfrak{u}, \\ \neg A & \text{if } \mathfrak{v}(X_i) = \mathfrak{f} \qquad (i = 1, 2, \cdots, n). \end{cases}$$

Then we have

$$\mathfrak{B}_i(\mathfrak{u}) = \mathfrak{v}(X_i) \qquad (i = 1, 2, \cdots, n).$$

Hence, $\mathfrak{A}(\mathfrak{B}_1(A), \cdots, \mathfrak{B}_n(A))$ is not a $\mathfrak{t}$-formula, because

$$\mathfrak{A}(\mathfrak{B}_1(\mathfrak{u}), \cdots, \mathfrak{B}_n(\mathfrak{u})) = \mathfrak{A}(\mathfrak{v}(X_1), \cdots, \mathfrak{v}(X_n)) \neq \mathfrak{t}, \text{ q.e.d.}$$

2.3. Let $\mathfrak{A}$ be a tautology which is not a $\mathfrak{t}$-formula. Let S be the system obtained from the intuitionistic propositional calculus by adjoining $\mathfrak{A}$ as an axiom schema. From the fact that $\mathfrak{A}$ is a tautology, we can see the fact that S is a subsystem of the classical propositional calculus. Accordingly, for our proof of Theorem 2, it is sufficient to prove the fact that there is a tautology, which contains only one propositional variable and is not a $\mathfrak{t}$-formula, and which is provable in S (by Corollary 2 of Lemma 1). But, by Lemma 2 the existence of such a tautology is clear. Then the proof of Theorem 2 is completed.

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2) By 'valuation' we mean here a valuation in the three-valued logic defined before.