218. The Separable Axiomatization of the Intermediate Propositional Systems S_n of Gödel

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In [3] Gödel introduced a series of many-valued propositional systems S_n, which is widely known and is quite frequently made use of when propositional systems are treated. And in our paper [6] we introduced two kinds of axiomatization for these S_n. But the *separation theorem* mentioned below does not hold on those axiomatized systems.

Separation Theorem. *A provable formula in the system can be proved using only the axioms for implication and those for the logical symbols actually appearing in the formula.*

We introduce, in this paper, another axiomatization for S_n and prove the separation theorem on them.

§1. Preliminaries. Definition 1.1. *S_n is a many-valued propositional system, whose values are integers $1, 2, \cdots, n$ and ω (ω is regarded greater than any positive integers), and whose sole designated value is 1. Logical operations $\supset, \wedge, \vee$, and $\neg$ are defined in S_n as follows*:

$$v_1 \supset v_2 = \begin{cases} 1 & \text{if } v_1 \geqq v_2, \\ v_2 & \text{otherwise}, \end{cases}$$
$$v_1 \wedge v_2 = \max(v_1, v_2),$$
$$v_1 \vee v_2 = \min(v_1, v_2),$$
$$\neg v = v \supset \omega.$$

An extension of S_n is LC of Dummett [2], in which values are defined to be all the positive integers and ω.

By $S \vdash A$, we mean that a formula A is provable (or valid) in the axiomatic system (or model) S. By $S + A_1 + \cdots + A_k$, we mean an axiomatic system obtained by adding the axiom schemes $A_1, \cdots, A_k$ to an axiomatic system S. If S_1 and S_2 are two systems axiomatic or defined by a model, we mean by $S_1 \supset S_2$ that the set of all provable or valid formulas of S_2 is included in that of S_1. And $S_1 \supset \subset S_2$ means that $S_1 \supset S_2$ and $S_2 \supset S_1$. If f is an assignment function of a model, we mean by $f(A)$ the value calculated for the formula A by the assignment f.

Lemma 1.2. $S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots \supset LC \supset LI$, *where LI is the intuitionistic system and S_1 coincides with the usual classical* *system.*

We omit the proof since the lemma can be easily proved. We call a system between the classical and the intuitionistic as *intermediate.*

§2. Former Axiomatizations. Definition 2.1. *We define some formulas as follows:*

$$X_n = \bigvee_{1 \leq i < j \leq n+1} (a_i \supset a_j) \wedge (a_j \supset a_i),$$
$$T_n = \bigvee_{1 \leq i \leq n+1} a_i,$$
$$F_n = \bigvee_{1 \leq i \leq n+1} \neg a_i,$$
$$Y_n = X_n \vee (T_n \wedge F_n),$$
$$R_n = a_n \vee (a_n \supset a_{n-1}) \vee \cdots \vee (a_2 \supset a_1) \vee \neg a_1,$$
$$Z = ((a_1 \supset a_2) \supset a_3) \supset (((a_2 \supset a_1) \supset a_3) \supset a_3),$$

where a_i's are propositional variables.

Dummett [2] obtained the

Lemma 2.2. $\boldsymbol{LC} \supset \subset \boldsymbol{LI} + Z.$

And we proved in [4] and [5] that the separation theorem holds on $\boldsymbol{LI} + Z$.

In [6] is proved the

Lemma 2.3. $\boldsymbol{S}_n \supset \subset \boldsymbol{LI} + Z + X_{n+1} + Y_n \supset \subset \boldsymbol{LI} + R_n.$

The separation theorem does not hold on these axiomatizations, since the newly added axiom schemes X_{n+1}, Y_n, and R_n contain logical symbols other than the implication.

§3. Separability. A formula is called an I (or C, or D, or N) formula if it contains only implication (or conjunction, or disjunction, or negation) as its logical symbols. An IC formula is a formula in which no logical symbols other than implication and conjunction are contained. An IC axiom is an axiom which is an IC formula. An IC theorem is a theorem which is an IC formula and is provable from IC axioms. An IC proof is a proof in which only IC axioms are used. A system is IC complete if the theorems which are IC formulas are IC theorems. Other combinations are defined similarly. A system is called *separable* if the separation theorem holds on it.

We proved in [4] the

Lemma 3.1. *If an intermediate propositional system satisfies the following* (1), (2), *and* (3), *it is separable.*

(1) *The system is constructed by adding some new I axioms to a separable intuitionistic propositional system.*

(2) *The system is I complete.*

(3) *There exist I formulas $F_i(a, b)$ $(i=1, \cdots, k)$ whose propositional variables are only a and b such that formulas of the forms*

$$D_i\colon a \vee b \supset F_i(a, b) \qquad (i=1, \cdots, k),$$

$$D_0\colon F_1(a, b) \supset (\cdots \supset (F_k(a, b) \supset a \vee b) \cdots),$$

are ID theorems.

The condition (3) can be weakened to (3′) below.

(3′) *The formulas D_i's and D_0 are theorems.*

Proof. Since $a \supset a \vee b$ and $b \subset a \vee b$ are intuitionistic theorems, we get I theorems $a \supset F_i(a, b)$ and $b \supset F_i(a, b)$ from the theorem D_i. Hence $a \vee b \supset F_i(a, b)$ is an ID theorem. And since

$$a \vee b \supset ((a \supset c) \supset ((b \supset c) \supset c))$$

is an intuitionistic theorem, we get an I theorem

$$F_1(a, b) \supset (\cdots \supset (F_k(a, b) \supset ((a \supset c) \supset ((b \supset c) \supset c))))$$

from D_0. Here we put c to be $a \vee b$, and from $a \supset a \vee b$ and $b \supset a \vee b$ we get the ID theorem D_0.

An example of the separable intuitionistic system is the system of Kleene [7]. And hereafter we mean by $\boldsymbol{LI}$ the intuitionistic system of Kleene or some other separable ones.

§ 4. $\boldsymbol{LP}_n$. Recently Nagata [8] defined a sequence of formulas P_n and systems $\boldsymbol{LP}_n$ as follows:

Definition 4.1.

$$P_1 = ((a_1 \supset a_0) \supset a_1) \supset a_1.$$
$$P_n = ((a_n \supset P_{n-1}) \supset a_n) \supset a_n.$$
$$\boldsymbol{LP}_n = \boldsymbol{LI} + P_n.$$

One of his results concerning this sequence is the

Lemma 4.2. $\boldsymbol{S}_n \supset \boldsymbol{LP}_n$, *but not* $\boldsymbol{S}_{n+1} \supset \boldsymbol{LP}_n$; *and*

$$\boldsymbol{S}_1 \supset \subset \boldsymbol{LP}_1 \supset \boldsymbol{LP}_2 \supset \cdots \supset \boldsymbol{LP}_n \supset \cdots \supset \boldsymbol{LI}.$$

Now we prove two lemmas concerning P_n.

Lemma 4.3. *Not* $\boldsymbol{LP}_n \supset \boldsymbol{S}_n$, *if* $n \geqq 2$.

Proof. Let $\boldsymbol{M}$ be a 5-valued system as follows:

$\supset$	1	2	3	4	ω
1	1	2	3	4	ω
2	1	1	3	4	ω
3	1	1	1	4	4
4	1	1	3	1	3
ω	1	1	1	1	1

$\wedge$	1	2	3	4	ω
1	1	2	3	4	ω
2	2	2	3	4	ω
3	3	3	3	ω	ω
4	4	4	ω	4	ω
ω	ω	ω	ω	ω	ω

$\vee$	1	2	3	4	ω
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	2	3
4	1	2	2	4	4
ω	1	2	3	4	ω

$\neg$	
1	ω
2	ω
3	4
4	3
ω	1

the value 1 is the designated value. Then it is easily seen that $\boldsymbol{M} \supset \boldsymbol{LI}$. $\boldsymbol{M} \vdash Z$ is not true since

$$((3 \supset 4) \supset 2) \supset (((4 \supset 3) \supset 2) \supset 2) = 2.$$

$P_1 \neq 1$ if and only if $a_1 = 2$ and $a_0 = 3, 4$, or ω, and then $P_1 = 2$. Hence if $n \geqq 2$, $P_n \neq 1$ if and only if $a_n = 2$ and $P_{n-1} = 3, 4$, or ω. But P_{n-1} only gets the value 1 or 2, if $n \geqq 2$. Hence $\boldsymbol{LP}_n \supset \boldsymbol{S}_n$ dose not hold if $n \geqq 2$.

Lemma 4.4. $\boldsymbol{LC} + P_n \vdash R_n$.

Proof. Let Q_n be a formula obtained from P_n by substituting a_0 by $a_1 \wedge \neg a_1$. Let f be an assignment function of $\boldsymbol{LC}$ assigning values $v_1, \cdots, v_n$ to propositional variables $a_1, \cdots, a_n$. Then $f(Q_n) = f(R_n)$ holds since $f(Q_n)$ and $f(R_n)$ are not 1 if and only if $1 < v_n < v_{n-1} < \cdots < v_1 < w$ and on that occasion they both gets the value v_n.

§ 5. New Axiomatization. Definition 5.1. $\boldsymbol{MP}_n = \boldsymbol{LC} + P_n$ (*that is*, $\boldsymbol{MP}_n = \boldsymbol{LI} + Z + P_n = \boldsymbol{LP}_n + Z$).

By 4.3, $\boldsymbol{LP}_n$ does not give an axiomatization for $\boldsymbol{S}_n$. But by 4.4, $\boldsymbol{MP}_n \supset \boldsymbol{S}_n$. And by 4.2, $\boldsymbol{S}_n \vdash P_n$, hence $\boldsymbol{S}_n \supset \boldsymbol{MP}_n$. So we have the

Theorem 5.2. $\boldsymbol{S}_n \supset \subset \boldsymbol{MP}_n$.

This $\boldsymbol{MP}_n$ is another axiomatization for $\boldsymbol{S}_n$. And the main purpose of this paper is to prove that the separation theorem holds on this $\boldsymbol{MP}_n$ if we take for $\boldsymbol{LI}$ a separable intuitionistic system such as that of Kleene [7].

Since we have the lemma 3.1, we only need to prove that $\boldsymbol{MP}_n$ satisfies the three conditions of 3.1. But the added axioms to $\boldsymbol{LI}$ are all I axioms, so (1) is satisfied. And

$$a \vee b \supset ((a \supset b) \supset b), \qquad a \vee b \supset ((b \supset a) \supset a),$$

and $((a \supset b) \supset b) \supset (((b \supset a) \supset a) \supset a \vee b)$ are (*ID*) theorems of LC, hence they are (*ID*) theorems of $\boldsymbol{MP}_n$ and (3) is satisfied. So we only need to prove that $\boldsymbol{MP}_n$ is I complete. Before we prove that, we must make some preparations.

An assignment function f is almost always considered with relation to some formulas, in other words, to a set of some propositional variables $\{a_1, \cdots, a_m\}$. Let v_i be the value assigned to a_i by f $(1 \leqq i \leqq m)$. By $V(f)$, we mean the set $\{v_1, \cdots, v_m\}$, and by $M_f(i)$ the i-th maximum value of $V(f)$ (we omit the subscript f if there occurs no confusion), and by $H(f)$ the number of different values in $V(f)$, and by f_k an assignment function which assigns to a_i the value 1 if $f(a_i) \leqq k$ and the value $f(a_i)$ otherwise.

Lemma 5.3. *If A is an I formula and $\boldsymbol{S}_n \vdash A$ and if f is an assignment function of* $\boldsymbol{LC}$ *such that $H(f) \leqq n$, or $H(f) = n+1$ and $V(f) \ni 1$, then $f(A) = 1$.*

Proof. The operation $\supset$ depends only on the relation $\geqq$ between the values, and $v_1 \supset v_2$ does not get a value other than 1 or v_2. So the calculation of $f(A)$ just goes as in $\boldsymbol{S}_n$ under the condition of the lemma.

Theorem 5.4. (*I* completeness.) *If A is an I formula and $\boldsymbol{S}_n \vdash A$, then A is provable in $\boldsymbol{MP}_n$ by using only I axioms.*

Proof. Let $b_1, \cdots, b_k$ be all the propositional variables appearing in the formula A. We assume, without loss of generality, that $k \geqq n+1$ since if not, we can take as A the I formula

$$(b_{k+1} \supset b_{k+1}) \supset (\cdots \supset ((b_{n+1} \supset b_{n+1}) \supset A) \cdots)$$

which is equivalent to A. Let ρ be a mapping function of $\{a_0, \cdots, a_n\}$ into $\{b_1, \cdots, b_k\}$. Then we define P_n^* to be the conjunction

$$\bigwedge_{\text{all } \rho} P_n(\rho(a_0), \cdots, \rho(a_n)),$$

where $P_n(\rho(a_0), \cdots, \rho(a_n))$ means the formula obtained by substituting a_i by $\rho(a_i)$ in P_n $(0 \leqq i \leqq n)$. Now let f be an assignment function of $\boldsymbol{LC}$. We prove that $f(P_n^* \supset A) = 1$.

If $H(f) \leqq n$, or if $H(f) = n+1$ and $V(f) \ni 1$, then $f(A) = 1$ by 5.3.

If $H(f) = n+1$ and $V(f) \not\ni 1$, or $H(f) > n+1$, $f(P_n^*) = M(n+1)$ since $f(P_n) \neq 1$ if and only if $1 < v_n < v_{n-1} < \cdots < v_0$ and on that occasion $f(P_n) = v_n$. Hence $f(P_n^* \supset A) = f_{M(n+1)}(A) = 1$ since $H(f_{M(n+1)}) = n+1$ and $V(f_{M(n+1)}) \ni 1$.

Since $(B_1 \wedge \cdots \wedge B_m \supset C) \equiv (B_1 \supset (B_2 \supset \cdots \supset (B_m \supset C) \cdots)$ is a theorem in $\boldsymbol{LC}$, $P_n^* \supset A$ can be transformed to an equivalent I formula of the form $Q_1 \supset (Q_2 \supset \cdots \supset (Q_m \supset A) \cdots)$, where each Q_i is a substituted form of P_n. And this transformed formula is provable in $\boldsymbol{LC}$. By the I completeness of $\boldsymbol{LC}$ (cf. [1] or [5]), it is provable by using only I axioms. On the other hand, each Q_i is provable in $\boldsymbol{MP}_n$ by using the I axiom P_n. Hence by the modus ponens, A is proved by I axioms only.

By this proof of I completeness of $\boldsymbol{MP}_n$, the separation theorem is proved on $\boldsymbol{MP}_n$.

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