253. *Boolean Multiplicative Closures. I*

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0. Introduction. We shall say that a multiplicative closure operator ∇ defined on a distributive lattice L with zero and unit is a *Boolean multiplicative closure operator* if any closed element under ∇ has a complement in L. Examples of Boolean multiplicative closure operators are the *possibility operator* defined by Gr. Moisil ([7], [8])[1] in (three-valued) Lukasiewicz algebras (see also [3] and [4]), and the operator D_1 defined by G. Epstein ([5], Definition 2) in Post algebras.

The aim of this note is to give a characterization of those distributive lattices (with zero and unit) that admits a Boolean multiplicative closure operator. In §1 we give the definitions and notations. In §2 we characterize additive-multiplicative closure operators by the set of their closed elements and in §3 we apply the results of §2 to solve our main problem. Finally, in §4 we show how some of the previous theorems can be extended to general multiplicative closure operators.

These results have some applications in the study of the lattice theory of many-valued logics. We were inspired in A. Monteiro's work on the ideal theory of (three-valued) Lukasiewicz algebras, that will be published elsewhere.

1. Definitions and notations. Let L be a distributive lattice with zero 0 and unit 1. we shall consider operators ∇ from L into L satisfying some of the following conditions:

C0) $\nabla 0=0$, C1) $x\leq \nabla x$, C2) $\nabla x=\nabla\nabla x$,
C3) If $x\leq y$, then $\nabla x\leq \nabla y$, C4) $\nabla(x\vee y)=\nabla x\vee \nabla y$,
C5) $\nabla(x\wedge y)=\nabla x\wedge \nabla y$.

If ∇ satisfies C1), C2), and C3) it is called a *closure operator* (see [9], [12], [2]), and we shall denote the set of all closure operators on L by $C(L)$.

If ∇ satisfies C1), C2), and C4), (or C1), C2), and C5)), it is called an additive closure operator ([10], [11], [6]) (or multiplicative closure operator, [1], [6]), and we shall denote by $\mathrm{Ca}(L)(\mathrm{Cm}(L))$ the set of additive (multiplicative) closure operators defined on L. It is clear

1) The references are contained in the second paper.

all that $\mathrm{Ca}(L)\subset\mathrm{C}(L)$ and that $\mathrm{Cm}(L)\subset\mathrm{C}(L)$. We define $\mathrm{Cam}(L)=\mathrm{Ca}(L)\cap\mathrm{Cm}(L)$.

If $A(L)$ is some class of operators defined on L, we denote by $\mathrm{Ao}(L)$ the set of all $\nabla\in A(L)$ that satisfies C0).

We shall say that a subset A of L is *lower relatively complete* if for all $x\in L$, the set $\{a\in A\colon\ x\leq a\}$ has a least element (i.e., an infimum belonging to A). We shall denote by $R(L)$ the class of all lower relatively complete subsets of L that contains the unit 1, and by $\mathrm{Ro}(L)$ the set of all $A\in R(L)$ such that $0\in A$.

It is clear that if $A\in R(L)$, and $x, y\in A$, then $x\wedge y\in A$. So, if $A\in R(L)$, A is a sublattice of L if and only if $x, y\in A$ implies that $x\vee y\in A$. We shall denote by $\mathrm{Rs}(L)(\mathrm{Ros}(L))$ the set of all sublattices of L belonging to $\mathrm{R}(L)(\mathrm{Ro}(L))$.

If ∇ is an operator of L into L, we say that $k\in L$ is *invariant* under ∇ if $\nabla k=k$, and we denote by $I(\nabla)$ the set of all invariant elements under ∇. If $\nabla\in C(L)$, it is usual to call the invariant elements under ∇ *closed* elements. The *range* of the operator ∇ is the set $\nabla L=\{x\colon x=\nabla y$ for some $y\in L\}$.

We reproduce here, for further reference, the following well known theorem ([9], [11]):

Theorem. 1.1. *If $\nabla\in C(L)$, then $\nabla(L)=I(\nabla)\in R(L)$, and for all $x\in L$ we have:*

$$\nabla x=\wedge\{k\in K\colon\quad x\leq k\} \tag{1}$$

where $K=I(\nabla)$. Conversely, if $K\in R(L)$, then (1) *defines a $\nabla\in C(L)$ and moreover, $K=I(\nabla)=\nabla(L)$. $\nabla\in\mathrm{Co}(L)$ if and only if $I(\nabla)\in\mathrm{Ro}(L)$; and $\nabla\in\mathrm{Ca}(L)$ if and only if $I(\nabla)\in\mathrm{Rs}(L)$.*

2. Additive-multiplicative closures. Let S be a sublattice of L. An ideal I of L is called an *S-ideal* in case that for any $x\in I$ there exits an element $s\in S$ such that $s\in I$ and $x\leq s$. It is easy to see that if I is an S-ideal, then the set $I_1=I\cap S$ is an ideal of the lattice S and that I is the ideal of L generated by I_1 (i.e., $I=\{x\in L\colon$ there exists $s\in I_1$ such that $x\leq s\}$). Conversely, if I_1 is an ideal of S, then the ideal I of L generated by I_1 is an S-ideal and $I_1=I\cap S$. So, *we have a one-to-one correspondence between the S-ideals of L and the ideals of the lattice S.* An ideal I of S is called *S-prime* in case that P is an S-ideal and P_1 is a prime ideal of the lattice S.

2.1. Lemma. *If I is an S-ideal contained in the prime ideal P of L, then there exists an S-prime ideal $\bar{P}$ such that $I\subset\bar{P}\subset P$.*

Proof. Setting $P_1=P\cap S$, we have that P_1 is a prime ideal of S. Hence, the ideal $\bar{P}$ generated in L by P_1 is an S-prime ideal, and obviously, $\bar{P}\subset P$. Furthermore, $I_1=I\cap S\subset P\cap S=P_1$, so $I\subset\bar{P}$.

The well known fact that in a distributive lattice every proper ideal is a set-intersection of prime ideals, allow us to prove the following:

2.2. Corollary. *Every proper S-ideal is a set-intersection of S-prime ideals.*

2.3. Corollary. *If $s \in S$ and $x \not\leq s$, then there exists an S-prime ideal P such that $s \in P$ and $x \notin P$.*

An S-ideal M of L that is not contained in any proper S-ideal different from M itself is called an *S-maximal ideal.* It is easy to see that M is a S-maximal ideal if and only if $M_1 = M \cap S$ is a maximal ideal of the lattice S. With a standard technique we can prove that:

2.4. Lemma. *If the sublattice S has a unit $1'$, then any S-ideal can be extended to an S-maximal ideal.*

It is also clear that *any S-maximal ideal is S-prime.*

2.5. Theorem. *If $\nabla \in \mathrm{Cam}(L)$ and $K = I(\nabla)$, then an ideal I of L is a K-ideal if and only if $x \in I$ implies that $\nabla x \in I$. In this case we have that $I_1 = I \cap K = \nabla I = \{\nabla x \colon x \in I\}$.*

Proof. By 1.1. we know that $K \in \mathrm{Rs}(L)$, so K is a sublattice of L. If I is a K-ideal and $x \in I$, then there exists a $k \in I \cap K$ such that $x \leq k$. Hence, $\nabla x \leq k$, and as k belongs to the ideal I, it follows that $\nabla x \in I$. On the other hand, if I is an ideal of L such that $x \in I$ implies that $\nabla x \in I$, it is obvious that I is a K-ideal. The second part of the theorem is an easy consequence of the first.

Q.E.D.

Our next theorem establish a characteristic property of the set of all closed elements under an additive-multiplicative closure operator on a distributive lattice:

2.6. Theorem. *$\nabla \in \mathrm{Cam}(L)$ if and only if $K = I(\nabla) \in \mathrm{Rs}(L)$ and every K-prime ideal is a prime ideal of L.*

Proof. Assume $\nabla \in \mathrm{Cam}(L)$. As $\mathrm{Cam}(L) \subset \mathrm{Ca}(L)$, from 1.1. follows that $K \in \mathrm{Rs}(L)$. Let P be a K-prime ideal. If $x \wedge y \in P$, then by 2.5., we have:

$$\nabla x \wedge \nabla y = \nabla(x \wedge y) \in P \cap K = P_1$$

but as P_1 is prime in K, $\nabla x \in P_1$ or $\nabla y \in P_1$, therefore, applying again 2.5., we get $x \in P$ or $y \in P$, and the necessity of the conditions is proved. Assume now that $K \in \mathrm{Rs}(L)$ and that any K-prime ideal is a prime ideal of L. By 1.1., we know that $\nabla \in \mathrm{Ca}(L)$, then we have:

(1) $$\nabla(x \wedge y) \leq \nabla x \wedge \nabla y.$$

So, to prove C5) we need to prove:

(2) $$\nabla x \wedge \nabla y \leq \nabla(x \wedge y).$$

Suppose that (2) is not true. Therefore, by 2.3., there exits a K-

prime ideal P such that:

(3) $\nabla(x\wedge y)\in P$ and (4) $\nabla x\wedge\nabla y\notin P$.

From (3) we have $x\wedge y\in P$, and as P is prime, $x\in P$ or $y\in P$. Applying 2.5. we get that $\nabla x\in P$ or $\nabla y\in P$, which contradicts (4), and the sufficiency of the conditions is proved. Q.E.D.

Now we are going to determine a particular class of K-prime ideals.

We say that a prime ideal P of L is a *minimal prime ideal* if it is a minimal element of the set of all prime ideals of L ordered by set-inclusion. It is well known that *any prime ideal of L contains a minimal prime ideal* (we suppose that L has a zero) and that *an ideal of L is a minimal prime ideal if and only if its complementary set is a maximal filter* (i.e., maximal dual ideal).

2.7. Theorem. *If $\nabla\in$ Coam(L) and $K=I(\nabla)$, then every minimal prime ideal of L is a K-prime ideal.*

Proof. Let P be a minimal prime ideal of L. It is clear that $P_1=P\cap K$ is a prime ideal of K, so we need to prove that P is a K-ideal, or, taking account of 2.5., that $x\in P$ implies that $\nabla x\in P$. Suppose that the last proposition is not true, that is, that there exists an element $x\in L$ such that:

(1) $x\in P$ and (2) $\nabla x\notin P$

Let F be the complementary set of P (with respect to L). (1) and (2) are equivalent respectively to:

(3) $x\notin F$ and (4) $\nabla x\in F$

Let $\bar{F}$ be the filter generated by the element x and the filter F. By (3) it follows that:

(5) $F\subset\bar{F}$ and $\bar{F}\neq F$

We are going to prove now:

(6) $\bar{F}\neq L$

or equivalently:

(7) $x\wedge f\neq 0$ for all $f\in F$.

To prove (7), suppose that there exists an element f_1 such that:

(8) $f_1\in F$ and (9) $x\wedge f_1=0$

From (9) follows:

(10) $\nabla x\wedge\nabla f_1=\nabla(x\wedge f_1)=0$

As F is a filter, (4), (8), and (10) imply that $0\in F$, or, what is the same, that $F=L$. Then, we would have $P=\phi$, but this is impossible by the hypothesis on P. Hence (6) is proved. But conditions (5) and (6) are incompatible, because F is a maximal filter, hence (1) and (2) cannot hold simultaneously. Q.E.D.