104. A Necessary and Sufficient Condition for a Semigroup to Have Identity Element

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(Comm. by Kinjirô KUNUGI, M.J.A., June 12, 1967)

Let S be a semigroup. Following E. S. Ljapin [3] we say that an element a of S is a *left magnifier* if S contains proper subset T such that

(1) $$aT=S \quad (T\subset S,\ T\neq S).$$

An element b in S is called a *right magnifier* if S contains a proper subset U such that

(2) $$Ub=S \quad (U\subset S,\ U\neq S).$$

An element a in S is called a *left* [*right*] *unit* of S if and only if

(3) $$aS=S \quad [Sa=S].$$

L. Rédei constructed an example for a semigroup with left unit and without left identity element (see [4]).

In this short note we prove the following result first published in Hungarian [2].

Theorem. *A semigroup S is a semigroup with identity element if and only if it contains at least one left unit which is not a left magnifier and at least one right unit which is not a right magnifier of S.*

Proof. Let us suppose that the semigroup S has an element a which is a left unit of S, but it is not a left magnifier of S. Furthermore, let b be a right unit of S, which is not a right magnifier in S. Then we have

(4) $$aS=S=Sb,$$

and this implies that there exist elements x,y in S such that

(5) $$ax=a \quad \text{and} \quad yb=b.$$

We show that the element x is a left unit, and y is a right unit of S. (4) and (5) imply

(6) $$axS=aS=S.$$

Hence we conclude that $xS=S$, because in the case $xS=T\subset S$ $(T\neq S)$ it follows that $aT=S$ with $T\subset S$, and the element a is a left magnifier of S, contrary to hypothesis. Therefore x is a left unit of S. Analogously can be proved that the element y is a right unit of S.

Next we show that a is a left cancellable element of S, that is $au=av$ implies $u=v$ for any elements u, v in S. If $au=av$ and

$u \neq v$, then $aT=S$ where $T=S\setminus v\subset S$, contradicting to the assumption that a is not a left magnifier. Therefore a is a left cancellable element of S. Analogously can be proved that b is a right cancellable element of S. (5) implies

$$axyb=axb \tag{7}$$

and

$$axyb=ayb. \tag{8}$$

The equations (7) and (8) imply

$$xy=x=y. \tag{9}$$

Thus the element $x=y$ at the same time is a left and a right unit of S, which implies that S is a semigroup with identity element (see [1] or [3]). The necessity is obvious.

Corollary 1. *A finite semigroup is a semigroup with identity if and only if it contains at least one left unit and at least one right unit.*

This is an easy consequence of our Theorem, because a finite semigroup contains no magnifying elements [3].

A cancellative semigroup also contains no magnifying elements, thus we have

Corollary 2. *A cancellative semigroup is a semigroup with identity if and only if it contains at least one left unit and at least one right unit.*

References

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