101. *On Spaces in Which Every Closed Set Is a G_δ*

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Let X be a topological space. In this note the property that every closed set in X is a G_δ is characterized in a manner which exhibits its relationship to the properties of countable metacompactness and countable paracompactness. In particular, it is shown that X is countably metacompact provided every closed set in X is a G_δ.

Unless explicit mention is made to the contrary, *no* separation axiom (e.g., the T_1-axiom) is assumed for the topological spaces under discussion. All terminology is consistent with that used in [2].

Theorem 1. *If X is a topological space, the following two statements are equivalent:*

(a) *every closed set F in X is a G_δ;*

(b) *whenever $F_1, F_2, \cdots$ is a decreasing sequence of closed sets in X, there exists a decreasing sequence $G_1, G_2, \cdots$ of open sets in X such that $\bigcap_{j=1}^{\infty} G_j = \bigcap_{i=1}^{\infty} F_i$ and $F_n \subset G_n$ for each positive integer n.*

Proof. Suppose (a) holds, that $F_1, F_2, \cdots$ is a decreasing sequence of closed sets in X, and that N is the set of positive integers. Then for each $i \in N$ there exists a decreasing sequence $H_1^i, H_2^i, \cdots$ of open sets in X such that $F_i = \bigcap_{j=1}^{\infty} H_j^i$. For each $j \in N$, let G_j be the open set $\bigcap_{i=1}^{j} H_j^i$. Then $G_1, G_2, \cdots$ is a decreasing sequence of open sets in X such that $F_n \subset G_n$ whenever $n \in N$. Furthermore, $\bigcap_{j=1}^{\infty} G_j = \bigcap_{j=1}^{\infty} \bigcap_{i=1}^{j} H_j^i = \bigcap_{j=1}^{\infty} \bigcap \{H_m^i : i, m \leqq j\}$, since $H_1^i \supset H_2^i \supset \cdots$ for each $i \in N$. Thus $\bigcap_{j=1}^{\infty} G_j = \bigcap \{H_j^i : i, j \in N\} = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} H_i^i = \bigcap_{i=1}^{\infty} F_i$ and so Statement (b) holds. That (b) implies (a) follows immediately by taking $F_i = F$ for each $i \in N$.

Corollary 1. *If X is a normal topological space, the following two statements are equivalent:*

(a) *every closed set F in X is a G_δ;*

(b) *whenever $F_1, F_2, \cdots$ is a decreasing sequence of closed sets in X, there exists a decreasing sequence $G_1, G_2, \cdots$ of open sets in X such that $\bigcap_{j=1}^{\infty} \bar{G}_j = \bigcap_{i=1}^{\infty} F_i$ and $F_n \subset G_n$ for each positive integer n.*

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Proof. Since X is normal, one may assume in the proof to Theorem 1 that $H_j^i \supset \bar{H}_{j+1}^i$ whenever i and j are positive integers. It then follows that $G_j \supset \bar{G}_{j+1}$ for each positive integer j, and so $\bigcap_{j=1}^{\infty} \bar{G}_j = \bigcap_{i=1}^{\infty} F_i$.

For convenience of reference, the following restatement of two theorems of F. Ishikawa [3] is included.

Lemma (F. Ishikawa.) *Let X be a topological space. Then X is*

(i) *countably metacompact if and only if, whenever $F_1, F_2, \cdots$ is a decreasing sequence of closed sets in X with $\bigcap_{i=1}^{\infty} F_i = \phi$, there exists a decreasing sequence $G_1, G_2, \cdots$ of open sets in X such that $\bigcap_{j=1}^{\infty} G_j = \phi$ and $F_n \subset G_n$ for each positive integer n.*

(ii) *countably paracompact if and only if, whenever $F_1, F_2, \cdots$ is a decreasing sequence of closed sets in X with $\bigcap_{i=1}^{\infty} F_i = \phi$, there exists a decreasing sequence $G_1, G_2, \cdots$ of open sets in X such that $\bigcap_{j=1}^{\infty} \bar{G}_j = \phi$ and $F_n \subset G_n$ for each positive integer n.*

Theorem 2. *Let X be a topological space.*

(i) *If every closed set in X is a G_δ, then X is countably metacompact.*

(ii) *If X is normal and every closed set in X is a G_δ, then X is countably paracompact.*[1]

Proof. The conditions which characterize countable metacompactness and countable paracompactness in the lemma are clearly implied by Statement (b) of Theorem 1 and of Corollary 1, respectively.

Note that the converses of both parts of Theorem 2 are invalid, even if X is a compact Hausdorff space. A counterexample is furnished by the space of ordinals less than or equal to the first uncountable ordinal (with the order topology).

The below example serves two purposes: first, to show that the normality hypothesis in Statement (ii) of Theorem 2 (and hence also in Corollary 1) can not be replaced with the assumption that X is a completely regular T_1-space; second, to describe a class of generalized compactness properties which are not consequences of the assumption that every closed set in X is a G_δ.

Example. *A completely regular T_1-space X such that*

(i) *every closed set in X is a G_δ, and hence*

(ii) *X is countably metacompact, but*

(iii) *X is not countably paracompact, and*

(iv) *X is not metaLindelöf (i.e., there exists an open cover of X which does not have a point-countable, open refinement).*

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1) First proved by C. H. Dowker [1, p. 221].

Construction. Let X consist of all points in the Euclidean xy-plane which lie above or on the x-axis L. Let the topology for X have as a base the set of all interiors of circles which are contained in $X-L$, together with all sets of the form $\{p\} \cup T$, where $p \in L$ and T is the interior of a circle in X which is tangent to L at p.

It is well-known that X is a completely regular T_1-space. E.g., see [4, p. 153]. Arguments given on p. 69 of [2] can be modified to show that X is neither countably paracompact nor metaLindelöf.

Let C be a closed set in X and let N be the set of positive integers. For each $p \in C \cap L$ and $n \in N$, let $T_n(p)$ be the union of $\{p\}$ and the interior of the circle in X which is tangent to L at p and has radius $1/n$. For each $p \in C-L$ and $n \in N$, let $D_n(p)$ be the intersection of $X-L$ with the interior of the circle which has radius $1/n$ and center at p. For each $n \in N$ let $G_n = [\cup\{T_n(p): p \in C \cap L\}] \cup [\cup\{D_n(p): p \in C-L\}]$. Then $G_1, G_2, \cdots$ is a decreasing sequence of open sets in X and $\bigcap_{n=1}^{\infty} G_n = C$. Thus C is a G_δ.

References

[1] C. H. Dowker: On Countably Paracompact Spaces. Canad. J. Math., **3**, 219-224 (1951).

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[3] Fumie Ishikawa: On Countably Paracompact Spaces. Proc. Japan Acad., **31**, 686-687 (1955).

[4] R. Vaidyanathaswamy: Treatise on set Topology. Part I. Madras: Indian Mathematical Society (1947).