140. *On the Harmonic Summability of Higher Order*

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§1. **Introduction.** Let $\{p_n\}$ be a given sequence of positive numbers and let $P_n=\sum_{k=0}^{n}p_k$. Given a series $\sum_{n=0}^{\infty}a_n$ with its partial sum s_n, if

$$\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}s_k\longrightarrow s \quad \text{as} \quad n\longrightarrow\infty,$$

the series $\sum_{n=0}^{\infty}a_n$ is said to be summable (N, p_n) to s. A regularity condition of the summability (N, p_n) is

$$p_n/P_n\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty.$$

(See Hardy [1, Theorem 16].) The summability $(N, 1/(n+1))$ is known as the harmonic summability. Concerning this summability, Iyengar [2] and Shaney [3] have defined the harmonic summability of higher order, independently. But, their definitions are different. The purpose of this paper is to investigate the relations between these methods of summation. Throughout this paper, p and p' denote positive integers. Iyengar's definition is as follows. Let $\alpha_{n,p}$ define by

$$\sum_{n=0}^{\infty}\alpha_{n,p}x^n\equiv\left\{\frac{1}{x}\log\frac{1}{1-x}\right\}^p=\left\{\sum_{n=0}^{\infty}\frac{x^n}{n+1}\right\}^p.$$

Then the summability (N, p_n) with $p_n=\alpha_{n,p}$ defines the summability (H, p).*) Of course, the summability $(H, 1)$ is the ordinary harmonic summability. If we use Lemma 4 stated below, we see that the regularity condition stated above is satisfied for the summability (H, p). Thus the summability (H, p) is regular. On the other, Shaney's definition is as follows. Let

$$\beta_{n,1}=(n+1)^{-1}$$

and, for $p\geqq 2$,

$$\beta_{n,p}=\left\{(n+1)\prod_{k=1}^{p-1}\log_k(n+\lambda)\right\}^{-1},$$

where $\log_1 x=\log x$, $\log_k x=\log(\log_{k-1}x)$, $k\geqq 2$, and λ is the least positive integer such that $\log_{p-1}\lambda>0$. Then the summability (N, p_n) with $p_n=\beta_{n,p}$ defines the summability (H', p). It should be noted that Shaney has used

*) Iyengar [2] has used the notation (N, p) in place of the notation (H, p) and later Varshney [4] has used this one.

$$\beta'_{n,p}=\left\{(n+1)\prod_{k=1}^{p-1}\log_k(n+1)\right\}^{-1}$$

in place of above $\beta_{n,p}$, but $\beta'_{0,p}$ has no meaning when $p\geqq 2$. The summability (H', p) is also regular.

Given two methods of summation P and Q, we write $P\Rightarrow Q$ if a series summable (P) is also summable (Q) and write $P\not\Rightarrow Q$ if there exists a series such that the series is summable (P) but not summable (Q). The results to be proved are as follows.

Theorem 1. $(H, p)\Rightarrow(H, p+1)$ *for a positive integer* p.

Theorem 2. $(H, p+1)\not\Rightarrow(H, p)$ *for a pasitive integer* p.

Theorem 3. $(H', p)\Rightarrow(H', 1)=(H, 1)$ *for a positive integer* $p\geqq 2$.

Theorem 4. $(H', p)\not\Rightarrow(H', p')$ *for positive integers* p *and* p', $p<p'$.

From these theorems we have the following two theorems.

Theorem 5. $(H', p)\Rightarrow(H, p')$ *for positive integers* p *and* p'.

Theorem 6. $(H, p)\not\Rightarrow(H', p')$ *for positive integers* p *and* p', $p'\geqq 2$.

§ 2. **Preliminary Lemmas. Lemma 1.** *If* (N, p_n) *and* (N, q_n) *are regular, then, in order that* $(N, p_n)\Rightarrow(N, q_n)$, *it is necessary and sufficient that*

$$\sum_{\nu=0}^{n}|k_\nu|P_{n-\nu}\leqq AQ_n, \tag{2.1}$$

where k_n *is defined by*

$$\sum_{n=0}^{\infty}k_nx^n\equiv\sum_{n=0}^{\infty}q_nx^n\Big/\sum_{n=0}^{\infty}p_nx^n=\sum_{n=0}^{\infty}Q_nx^n\Big/\sum_{n=0}^{\infty}P_nx^n,$$

and A *is a constant independent on* n, *and that*

$$k_n/Q_n\to 0 \quad \text{as} \quad n\to\infty.$$

This is a theorem due to M. Riesz. (See Hardy [1, *Theorem* 19].)

Lemma 2. *If* (N, p_n) *and* (N, q_n) *are regular, and if there does not exist a constant* A' *such that*

$$P_n\leqq A'Q_n, \qquad n=1, 2, 3, \cdots,$$

then $(N, p_n)\not\Rightarrow(N, q_n)$.

Proof. If $(N, p_n)\Rightarrow(N, q_n)$, we have, from (2.1), a constant A such that

$$P_n\leqq AQ_n, \qquad n=1, 2, 3, \cdots$$

This contradictes our assumption.

Lemma 3. *If (i)* (N, p_n) *and* (N, q_n) *are regular, (ii)* p_n *satisfies*

$$p_n>0, \qquad p_{n+1}/p_n\geqq p_n/p_{n-1}, \qquad n>0, \tag{2.2}$$

(iii) $q_n>0$ *and (iv)*

$$q_n/q_{n-1}\geqq p_n/p_{n-1}, \qquad n>n_0,$$

then $(N, p_n)\Rightarrow(N, q_n)$.

This is a theorem due to Hardy. (See Hardy [1, Theorem 23 and p. 91].)

Lemma 4. *Let p be a positive integer and let $\alpha_{n,p}$ and $\gamma_{n,p}$ be defined by*

$$\sum_{n=0}^{\infty} \alpha_{n,p} x^n \equiv \left\{\frac{1}{x} \log \frac{1}{1-x}\right\}^p \quad \textit{and} \quad \left(1-\sum_{n=1}^{\infty} \gamma_{n,p} x^n\right)\left(\sum_{n=0}^{\infty} \alpha_{n,p} x^n\right) \equiv 1.$$

Then we have

$$\alpha_{n,p} \sim \frac{1}{n}(\log n)^{p-1}, \tag{2.3}$$

$$\sum_{k=0}^{n} \alpha_{k,p} \sim (\log n)^p \tag{2.4}$$

and

$$\gamma_{n,p} = O(1/n(\log n)^{p+1}). \tag{2.5}$$

Proof. (2.5) is due to Iyengar [2, Lemma 1]. To prove (2.3), if we use the relations

$$\alpha_{n,p} = \frac{p}{n+p} \sum_{k=0}^{n} \alpha_{k,p-1} \quad \text{and} \quad \alpha_{n,1} = \frac{1}{n+1}, \tag{2.6}$$

then (2.3) is easily proved by induction. The first relation of (2.6) is obtained as follows. Since

$$\frac{d}{dx}\left(\log \frac{1}{1-x}\right)^p = p\left(\sum_{n=0}^{\infty} \alpha_{n,p-1} x^n\right)\left(\sum_{n=0}^{\infty} x^n\right)$$
$$= p \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \alpha_{k,p-1}\right) x^{n+p-1},$$

we see, by termwise integration,

$$\sum_{n=0}^{\infty} \alpha_{n,p} x^n = \sum_{n=0}^{\infty} \left(\frac{p}{n+p} \sum_{k=0}^{n} \alpha_{k,p-1}\right) x^n.$$

Thus we have (2.6). Concerning (2.4), since $(\log x)^{p-1}/x$ is monotone decreasing on the interval $[x_0, \infty)$, x_0 being a constant, we have, using (2.3),

$$\sum_{k=0}^{n} \alpha_{k,p} \sim \sum_{k=0}^{n} \frac{(\log (k+1))^{p-1}}{k+1} \sim \int_1^n \frac{(\log x)^{p-1}}{x} dx = (\log n)^p,$$

which is the required result.

§3. Proof of Theorems. 3.1. Proof of Theorem 1. We set

$$p(x) \equiv \sum_{n=0}^{\infty} \alpha_{n,p} x^n \equiv \left\{\frac{1}{x} \log \frac{1}{1-x}\right\}^p, \quad q(x) \equiv \sum_{n=0}^{\infty} \alpha_{n,p+1} x^n \equiv \left\{\frac{1}{x} \log \frac{1}{1-x}\right\}^{p+1}$$

and

$$k(x) \equiv \frac{q(x)}{p(x)} = \frac{1}{x} \log \frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{x^n}{n+1} \equiv \sum_{n=0}^{\infty} k_n x^n.$$

Thus

$$k_n = \frac{1}{n+1} > 0 \qquad \text{for all } n.$$

Hence, by Lemma 1, we have

$$(H, p) \Rightarrow (H, p+1).$$

3.2. Proof of Theorem 2. Let us put

$$p_n=\alpha_{n,p+1} \quad \text{and} \quad q_n=\alpha_{n,p}.$$

Then, by Lemma 3,

$$P_n=\sum_{k=0}^{n} p_k \sim (\log n)^{p+1} \quad \text{and} \quad Q_n=\sum_{k=0}^{n} q_k \sim (\log n)^p.$$

Thus, by Lemma 2, we have

$$(H, p+1) \not\Rightarrow (H, p).$$

3.3. Proof of Theorem 3. Let us put

$$p_n=\beta_{n,p} \quad \text{and} \quad q_n=\frac{1}{n+1}.$$

To prove the theorem, we have to verify that the conditions of Lemma 3 are satisfied. Except the second condition of (2.2), the other are trivially satisfied. Hence we shall prove the second condition of (2.2). Since the function $-\log x$ is convex, we have

$$\left(\frac{1}{n+1}\right)^2<\frac{1}{n(n+2)};$$

since the function $-\log_2 x$ is convex, we have

$$\left(\frac{1}{\log (n+\lambda)}\right)^2<\frac{1}{\log (n-1+\lambda) \log (n+1+\lambda)};$$

and so we have

$$\left(\frac{1}{\log_{p-1} (n+\lambda)}\right)^2<\frac{1}{\log_{p-1} (n-1+\lambda) \log_{p-1} (n+1+\lambda)}.$$

Thus we have

$$p_{n-1}p_{n+1}>p_n^2, \text{ i.e., } \frac{p_{n+1}}{p_n}>\frac{p_n}{p_{n-1}},$$

which is the required result.

3.4. Proof of Theorem 4. Let us put

$$p_n=\beta_{n,p} \quad \text{and} \quad q_n=\beta_{n,p'}.$$

Then we have

$$P_n=\sum_{k=0}^{n} p_k \sim \log_p n \quad \text{and} \quad Q_n=\sum_{k=0}^{n} q_k \sim \log_{p'} n.$$

Since $p<p'$, we see, using Lemma 2,

$$(H', p) \not\Rightarrow (H', p').$$

References

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