30. On the Representations of $SL(3, C)$. II

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1. Considering the interwinning operators of the representations $\mathfrak{R}(\chi)=\{T^{\chi}, \mathfrak{D}_{\chi}\}(\chi=(\lambda_1, \mu_1; \lambda_2, \mu_2))$ of $G=SL(3, C)$ discussed in the part I of this work [1], we can obtain the equivalence relations and semi-reduciblity of these representations in the present paper. We shall consider three cases separately: (i) Neither one of pairs $(\lambda_k, \mu_k)(k=1, 2)$ nor $(\lambda_1+\lambda_2, \mu_1+\mu_2)$ is a pair of integers. (ii) Only one of them is a pair of integers. (iii) All of them are pairs of integers. We shall consider unitary representations in the next paper.

During the preparation of this paper, the author found that Zhelobenko [2] published important results which concern this part of the present works. In this paper we shall consider representations in concrete form.

2. Let $(\lambda_1, \mu_1)=(l_1, m_1)$ be a pair of positive integers, then in the space $\mathfrak{D}_{\chi}$ there exists an invariant subspace $\mathcal{E}_{\chi}^{1}$ which is the linear subspace of polynomials in z_1 and $z_1z_2-z_3$ of degree (l_1-1, m_1-1) with coefficients $a_{pq}(z_2, z_3)$ which are such C^{∞}-functions that $z_2^{(\lambda_2-1,\mu_2-1)}a_{pq}(1/z_2, z_3/z_2)$ and $z_3^{(\lambda_2-1,\mu_2-1)}a_{pq}(z_2/(z_3, 1/z_3)$ are again C^{∞}-functions.

Then we have

Theorem 1. (1°) *If neither one of pairs (λ_k, μ_k) nor $(\lambda_1+\lambda_2, \mu_1+\mu_2)$ is a pair of integers of the same signature, then the representation $\mathfrak{R}(\chi)$ is completely irreducible.*

(2°) *If $(\lambda_1, \mu_1)=(l_1, m_1)$ is a pair of positive integers and (λ_2, μ_2) is not a pair of integers of the same signature, then the representation $\{T^{\chi}, \mathcal{E}_{\chi}^{1}\}$ is completely irreducible.*

We omit the proof of this theorem. The principal purpose of the present paper is to prove the following theorem in section 3, 4, and 5:

Theorem 2. *The representations $\mathfrak{R}(\chi)$, $\mathfrak{R}(\chi')$ of G are equivalent or partially equivalent if and only if $\chi'=\chi^{s}$ for some $s\in W$.*

3. *Case* (*i*) (*non-degenerate case*). There are six intertwinning operators from $\mathfrak{R}(\chi)$ to $\mathfrak{R}(\chi^{s})$:

$$A_1\varphi(z)=\gamma(\lambda_1, \mu_1)\int z_1'^{(-\lambda_1-1,-\mu_1-1)}\varphi(z_1'+z_1, z_2, z_3)dz_1',$$

$$A_2\varphi(z)=\gamma(\lambda_2, \mu_2)\int z_2'^{(-\lambda_2-1,-\mu_2-1)}\varphi(z_1, z_2'+z_2, z_1z_2-z_3)dz_2',$$

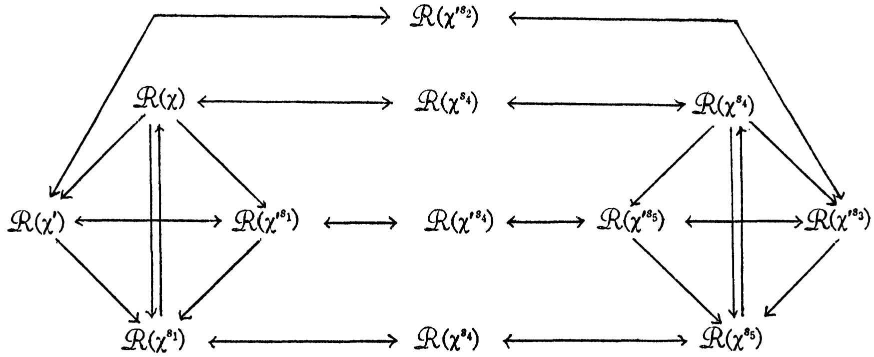
A_2A_1, A_1A_2, $A_1A_2A_1=A_2A_1A_2$ and the identity operator. A_1 and A_2 are bijective, therefore we obtain the following equivalence relations of completely irreducible representations $\mathscr{R}(\chi^{s_i})(i=0, \cdots, 5)$(the arrow $\longleftrightarrow$ means equivalence):

$$\mathscr{R}(\chi)\overset{A_1}{\longleftrightarrow}\mathscr{R}(\chi^{s_1})\overset{A_2}{\longleftrightarrow}\mathscr{R}(\chi^{s_4})\overset{A_1}{\longleftrightarrow}\mathscr{R}(\chi^{s_5})$$
$$\overset{A_2}{\longleftrightarrow}\mathscr{R}(\chi^{s_3})\overset{A_1}{\longleftrightarrow}\mathscr{R}(\chi^{s_2})\overset{A_2}{\longleftrightarrow}\mathscr{R}(\chi).$$

4. *Case* (*ii*) (*degenerate case* (1)). We may start with $\mathscr{R}(\chi)$ where $(\lambda_1, \mu_1)=(l_1, m_1)$ is composed of positive integers. The intertwinning operators from $\mathscr{R}(\chi)$ are A_1, A_2 of 3, and

$$A_1'\varphi(z)=\int\delta^{(l_1,0)}(z_1')\varphi(z_1'z)dz',$$
$$A_1''\varphi(z)=\int\delta^{(0,m_1)}(z_1')\varphi(z_1'z)dz',$$

and the possible products of them and the identity. By these operators we obtain the following relations (the arrow $\rightleftarrows$ means partial equivalence):

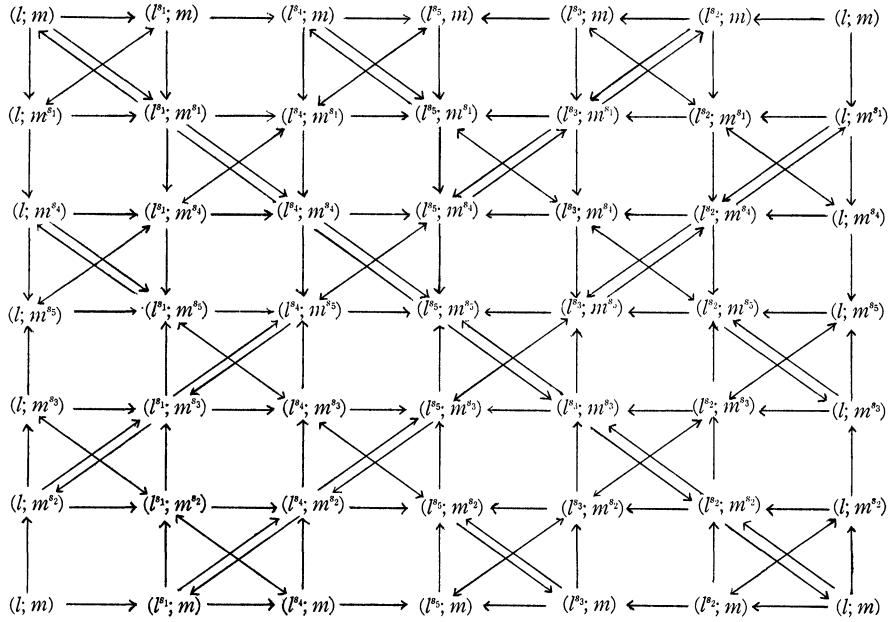


In the diagram the character χ' means $(-l_1, m_1; l_1+\lambda_2, \mu_2)$ and the representation $\mathscr{R}(\chi')$ is completely irreducible. As for $\mathscr{R}(\chi^{s_1})$, there exists the invariant subspace $\mathscr{F}^1_{\chi^{s_1}}$ of all such functions $\varphi(z)$ of $\mathscr{D}_{\chi^{s_1}}$ that $\int z_1'^{(l_1-1,m_1-1)}\varphi(z_1'z)dz_1'=0$.

Proposition. *Representations* $\{T^\chi, \mathscr{D}_\chi/\mathscr{E}^1_\chi\}$, $\{T^{\chi^{s_1}}, \mathscr{F}^1_{\chi^{s_1}}\}$ *and* $\mathscr{R}(\chi')$ *are equivalet, and* $\{T^\chi, \mathscr{E}^1_\chi\}$ *and* $\{T^{\chi^{s_1}}, \mathscr{D}_{\chi^{s_1}}/\mathscr{F}^1_{\chi^{s_1}}\}$ *are equivalent.*

The diagram and this proposition show that $\mathscr{R}(\chi^{s_i})$ are partially equivalent to $\mathscr{R}(\chi)$, more precisely, every (completely irreducible) sub-representation and factor-representation of $\mathscr{R}(\chi^{s_i})$ are equivalent to the sub-representation or factor-representation of $\mathscr{R}(\chi)$.

5. *Case* (*iii*) (*degenerate case* (2)). We may start with $\mathscr{R}(\chi)$ where all of l_k, m_k are positive integers. The intertwinning operators from $\mathscr{R}(\chi)$ are A_1, A_1', and A_1'' in 4 and analogous operators A_1, A_2', and A_2'' and their possible products. By these operators we obtain



the diagram on the next page.

In the space $\mathcal{D}_\chi$ there exist the invariant subspaces $\mathcal{E}_\chi^1$, $\mathcal{E}_\chi^2$ (the latter is defined analogously to $\mathcal{E}_\chi^1$), $\mathcal{A}_\chi = \mathcal{E}_\chi^1 \cup \mathcal{E}_\chi^2$, and $\mathcal{E}_\chi = \mathcal{E}_\chi^1 \cap \mathcal{E}_\chi^2$. The representations $\mathcal{R}(\chi^{s_i})$ are partially equivalent to $\mathcal{R}(\chi)$, that is, factor-representations $\{T^\chi, \mathcal{D}_\chi/\mathcal{A}_\chi\}$, $\{T^\chi, \mathcal{E}_\chi^1/\mathcal{E}_\chi\}$ and $\{T^\chi, \mathcal{E}_\chi^2/\mathcal{E}_\chi\}$ and sub-representation $\{E^\chi, \mathcal{E}_\chi\}$(finite-dimensional representation) are completely irreducible and sub-representation and factor-representations of $\mathcal{R}(\chi^{s_i})$ are equivalent to one of those representations cited above. Other representations in the diagram are contained in $\mathcal{R}(\chi^{s_i})$ in such a way that, for instance, $\{T^{(l^{s_5}, m)}, \mathcal{D}_{(l^{s_5}, m)}\}$ is equivalent with $\{T^\chi, \mathcal{D}_\chi/\mathcal{A}_\chi\}$ and $\{T^{\chi^{s_5}}, \mathcal{F}_{\chi^{s_5}}\}$, where $\mathcal{F}_{\chi^{s_5}} = \mathcal{F}_{\chi^{s_5}}^1 \cap \mathcal{F}_{\chi^{s_5}}^2$.

The author wishes to publish the details of these facts in another paper.

References

[1] M. Tsuchikawa: On the representations of $SL(3, C)$. I. Proc. Japan Acad., **43**, 852–855 (1967).

[2] D. P. Zhelobenko: Symmetries on the class of elementary representations of complex semi-simple Lie groups (in Russian). Functional Analysis. A.N., U.S.S.R., **1**(2), 15–38 (1967).