29. *Remarks on Countable Paracompactness*

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Introduction. This note consists of 2 parts. 1. An observation that every regular normal space is countably paracompact if and only if every T_4 space is countably paracompact. 2. It is proved that a generalized F_σ, α-countably paracompact subset is closed in a T_4 space. α-countably paracompact subsets were introduced in [2].

Dowker's problem. Dowker [5, 221] gave an example of a normal space that was not countably paracompact, and he raised the question as to the existence of T_4 spaces that were not countably paracompact. In that Dowker's example is T_0 and not T_1, it is not regular. We use a result of M. H. Stone to prove the following theorem.

Theorem 1. *Every normal regular topological space is countably paracompact if and only if every T_4 space is countably paracompact.*

Proof. Let $(X, \mathcal{T})$ be a regular normal space. For $x \in X$, let $\langle x \rangle$ be the intersection of all open and closed sets containing $[x]$ as a subset. We define equivalence classes on $X \times X$ by saying that $(x, y) \in R$ if and only if $\langle x \rangle = \langle y \rangle$. Then by Stone's theorem X/R with the quotient topology $\mathcal{T}_R$ is a T_0 space and $(X, \mathcal{T})$ and $(X/R, \mathcal{T}_R)$ are lattice equivalent and one is normal (regular) then the other is normal (regular). So $(X/R, \mathcal{T}_R)$ is T_4. Let $\{V_n\}$ be a countable open cover of $(X, \mathcal{T})$. Let $\phi(V_n)$ be the image of V_n in $\{X/R, \mathcal{T}_R\}$. $\{\phi(V_n)\}$ is clearly a cover of X/R. Let $(X/R, \mathcal{T}_R)$ be countably paracompact, then $\{\phi(V_n)\}$ has a point finite countable open refinement $\{\phi(W_n)\}$. $\{W_n\}$ is a cover of X since if $x \in T \in \mathcal{T}, y \in T$ if $y \in \langle x \rangle$. Clearly $\{W_n\}$ is a point finite open refinement of $\{V_n\}$ and $(X, \mathcal{T})$ is countably metacompact and by a theorem of Dowker [5, 220] is countably paracompact.

α-countably paracompact subsets.

Definition 1. *A subset M of a topological space $(X, \mathcal{T})$ is α-countably paracompact if every countably $\mathcal{T}$-open cover $\{V_n\}$ has a $\mathcal{T}$-open refinement (covers X) which is locally finite with respect to all points of X.*

In a previous paper [1], it was proved that, in topological spaces such that every point is the intersection of a countable number of closed neighborhoods, α-countably paracompact subsets are closed. We wish to prove a related result, using the concept of a generalized F_σ. See Corson and Michael [4].

Theorem 2. *A generalized F_σ α-countably paracompact subset of a T_4 space $(X, \mathcal{T})$ is closed.*

Proof. Let M be a generalized F_σ, α-countably paracompact proper subset of a T_4 space. Let $x \notin M$. By the T_1 property of $(X, \mathcal{T})$, there exists an open set V such that $x \notin V$ and $M \subset V$; there exists an F_σ set $H = \bigcup F_k$ such that $M \subset H \subset V$ and each F_k is closed. By the normality of $(X, \mathcal{T})$ there exists a family of open sets $\{G_k\}$ such that $F_k \subset G_k \subset \bar{G}_k \subset V$. By the α-countable paracompactness of M, there is an open locally finite family $\{W_k\}$ covering M such that $W_k \subset G_k$. $M \subset \bigcup \bar{W}_k \subset \sim [x]$. So M is closed.

Corollary 1. *A generalized F_σ countably compact subset of a T_4 space $(X, \mathcal{T})$ is closed.*

It follows that in a perfectly normal compact T_1 space the countably compact and the compact subsets are identical.

References

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