26. Oscillatory Property of Certain Non-linear Ordinary Differential Equations

By Hiroshi ONOSE

Department of Mathematics, Ibaraki University

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1. **Statement of theorems.** Recently, Kartsatos [2] proved that certain differential equations of the form

$$x''+f(t)g(x, x')=0 \quad \text{or} \quad x^{(2n)}+f(t)g(x)=0$$

can have only oscillatory solutions. Looking into the proofs in [2] closely, we see that the argument used there may be applied equally well to equations of the following more general form:

$$(1) \qquad x^{(2n)}+f(t)g(x, x', \cdots, x^{(2n-1)})=0.$$

We shall prove in this paper the following theorems, where all functions considered are real-valued and continuous on their domains.

Theorem 1. *Suppose that the differential equation* (1) *satisfies the following hypotheses:*

(α) *f is a positive function defined on the interval $I=[t_0, +\infty)$ with $t_0\geqslant 0$ and $\int_{t_0}^{+\infty}f(t)dt=+\infty$;*

(β) *g is defined on R^{2n}; $\operatorname{sgn} g(x_1, x_2, \cdots, x_{2n})=\operatorname{sgn} x_1$ for any $(x_1, x_2, \cdots, x_{2n})\in R^{2n}$; and $g(\lambda x_1, \lambda x_2, \cdots, \lambda x_{2n})=\lambda^{2p+1}g(x_1, x_2, \cdots, x_{2n})$ for any $(x_1, x_2, \cdots, x_{2n})\in R^{2n}$, any $\lambda\in R$ and some non-negative integer p. Then, every solution of* (1) *on the interval I is oscillatory.*

Theorem 2. *Suppose that the equation* (1) *satisfies* (α) *and the following:*

(γ) *g is defined on R^{2n}; $\operatorname{sgn} g(x_1, x_2, \cdots, x_{2n})=\operatorname{sgn} x_1$ for any $(x_1, x_2, \cdots, x_{2n})\in R^{2n}$; $g(-x_1, -x_2, \cdots, -x_{2n})=-g(x_1, x_2, \cdots, x_{2n})$ for any $(x_1, x_2, \cdots, x_{2n})\in R^{2n}$; and for any $2\leqslant k\leqslant 2n-1$ and any $c\geqslant 0$, the function $g(x_1, x_2, \cdots, x_{2n})$ has a definite limit $G(k, c)$, which is positive or $+\infty$, as $x_1\to+\infty, \cdots, x_{k-1}\to+\infty, x_k\to c, x_{k+1}\to 0, \cdots, x_{2n}\to 0$. Then, every solution of* (1) *on I is oscillatory.*

We would like to remark that Kartsatos [2] proved Theorem 1 in the case $n=1$ and Theorem 2 when the function g depends only on the variable x_1.

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2. **Proof of theorems.** First we shall prove the following elementary but useful

Lemma. *Let φ be a $2n$-times continuously differentiable func-*

tion defined on the interval $I=[t_0, +\infty)$. *If* $\varphi>0$ *and* $\varphi^{(2n)}<0$ *on the interval* I, *then*

$$\lim_{t\to+\infty}\frac{\varphi^{(k)}(t)}{\varphi(t)}=0 \qquad \textit{for } 1\leqslant k\leqslant 2n-1.$$

Proof of lemma. As $\varphi^{(2n)}<0$ on the interval I, $\varphi^{(2n-1)}$ is decreasing on I, so that $\varphi^{(2n-1)}(t)$ has a limit, finite or $-\infty$, as $t\to+\infty$. We denote this limit by $\varphi^{(2n-1)}(\infty)$. We shall use similar notations in what follows.

Case 1: If $\varphi^{(2n-1)}(\infty)<0$, then we see easily by integration that $\varphi^{(2n-2)}(t)\to-\infty$ as $t\to+\infty$, and by the same argument as above $\varphi^{(k)}(t)\to-\infty$ as $t\to+\infty$ for $1\leqslant k\leqslant 2n-3$. And finally $\varphi(t)\to-\infty$ as $t\to+\infty$, which contradicts the hypothesis $\varphi>0$ on I.

Case 2: If $\varphi^{(2n-1)}(\infty)>0$, then $\varphi^{(2n-2)}(t)\to+\infty$ as $t\to+\infty$ and so $\varphi^{(k)}(t)\to+\infty$ as $t\to+\infty$ for $0\leqslant k\leqslant 2n-2$. In this case, we have $\varphi^{(k)}(t)/\varphi(t)\to 0$ as $t\to+\infty$ for $1\leqslant k\leqslant 2n-1$. In fact, this is trivial for $k=2n-1$, because $\varphi^{(2n-1)}(\infty)$ is finite. For $1\leqslant k\leqslant 2n-2$, it can be shown by using l'Hospital's rule.

Case 3: If $\varphi^{(2n-1)}(\infty)=0$, then $\varphi^{(2n-1)}>0$ on I and therefore $\varphi^{(2n-2)}$ is increasing on I. Thus, $\varphi^{(2n-2)}(\infty)$ exists and is finite or $+\infty$. If $\varphi^{(2n-2)}(\infty)<0$, then $\varphi^{(k)}(t)\to-\infty$ for $0\leqslant k\leqslant 2n-3$ and we get a contradiction. If $\varphi^{(2n-2)}(\infty)>0$, then $\varphi^{(k)}(t)\to+\infty$ as $t\to+\infty$ for $0\leqslant k\leqslant 2n-3$. In this case, $\varphi^{(k)}(t)/\varphi(t)\to 0$ as $t\to+\infty$ by the above mentioned arguments, so that the lemma is true. If $\varphi^{(2n-2)}(\infty)=0$, then $\varphi^{(2n-2)}<0$ on I and so $\varphi^{(2n-3)}$ is decreasing on I. Here we may also have three cases: $\varphi^{(2n-3)}(\infty)>0$, $\varphi^{(2n-3)}(\infty)<0$ or $\varphi^{(2n-3)}(\infty)=0$. In the first case, the lemma is true. The second case never happens, because it will lead to a contradiction. So we have only to dispose of the third case. Then, $\varphi^{(2n-3)}>0$ on I and we again have to examine the limit $\varphi^{(2n-4)}(\infty)$.

Repeating similar arguments, we shall have only the following case that remains to be unsettled:

$$(-1)^k\varphi^{(k)}<0 \quad\text{and}\quad \varphi^{(k)}(\infty)=0 \qquad \text{for } 1\leqslant k\leqslant 2n-1.$$

In this case, φ is increasing. As $\varphi>0$ by hypothesis, we have $\varphi(\infty)>0$ and therefore $\varphi^{(k)}(t)/\varphi(t)\to\varphi^{(k)}(\infty)/\varphi(\infty)=0$ as $t\to+\infty$ for $1\leqslant k\leqslant 2n-1$. Hence our lemma is completely proved. Q.E.D.

In the above proof, we have shown the following

Corollary. *If* $\varphi>0$ *and* $\varphi^{(2n)}<0$ *on* I, *then each* $\varphi^{(k)}$ *with* $0\leqslant k\leqslant 2n-1$ *has a definite limit as* $t\to+\infty$. *If we denote by* $\varphi^{(k)}(\infty)$ *these limits, then there happen only the following cases*: $\varphi(\infty)=\cdots=\varphi^{(k-1)}(\infty)=+\infty, \varphi^{(k)}(\infty)=c\geqslant 0, \varphi^{(k+1)}(\infty)=\cdots=\varphi^{(2n-1)}(\infty)=0$ *with* $0\leqslant k\leqslant 2n-1$, *where* $c>0$ *in case* $k=0$.

Proof of Theorem 1. Assume on the contrary that there exists a solution $x(t)$ of the equation (1) which does not oscillate on some interval $[t_1, +\infty)$ with $t_1 \geqslant t_0$. This solution x can be supposed to be positive on the above interval. By (1), we have $x^{(2n)} = -f(t) \times g(x, x', \cdots, x^{(2n-1)})$ and so our hypothese simply that $x^{(2n)} < 0$ for all $t \geqslant t_1$. Using the lemma, we have

$$\lim_{t\to+\infty} \frac{x^{(k)}(t)}{x(t)} = 0 \qquad \text{for } 1 \leqslant k \leqslant 2n-1.$$

So, for any $\varepsilon > 0$, there exists $T \geqslant t_0$ such that

$$\left| g\left(1, \frac{x'}{x}, \cdots, \frac{x^{(2n-1)}}{x}\right) - g(1, 0, \cdots, 0) \right| < \varepsilon \qquad \text{for } t \geqslant T.$$

As we have sgn $g(1, 0, \cdots, 0) = \text{sgn } 1$, so $g(1, 0, \cdots, 0) > 0$ and therefore we may assume $0 < \varepsilon < g(1, 0, \cdots, 0)$. As we see from the proof of the lemma, both x' and $x^{(2n-1)}$ are non-negative for sufficiently large t. Thus we may assume that $x(t) > c_0$, $x'(t) \geqslant 0$, $x^{(2n-1)}(t) \geqslant 0$ and $x^{(2n)}(t) < 0$ for all $t \geqslant t_2$, where $t_2 \geqslant \max(t_1, T)$ and c_0 is some positive constant.

If we put $y = x^{(2n-1)}/x$, then we have

$$y'(t) = \frac{x^{(2n)}(t)}{x(t)} - \frac{x'(t)x^{(2n-1)}(t)}{x(t)^2} \leqslant \frac{x^{(2n)}(t)}{x(t)}$$

for $t \geqslant t_2$. Integrating this inequality over the interval $[t_2, t]$ and using the equation (1), we have

$$\begin{aligned} \frac{x^{(2n-1)}(t)}{x(t)} - \frac{x^{(2n-1)}(t_2)}{x(t_2)} &\leqslant -\int_{t_2}^{t} \frac{f(s)g(x, x', \cdots, x^{(2n-1)})}{x(s)} ds \\ &= -\int_{t_2}^{t} f(s)x^{2p}(s)g\left(1, \frac{x'}{x}, \cdots, \frac{x^{(2n-1)}}{x}\right) ds \\ &\leqslant -[g(1, 0, \cdots, 0) - \varepsilon] c_0^{2p} \int_{t_2}^{t} f(s) ds, \end{aligned}$$

which implies a contradiction; in fact, the last member tends to $-\infty$ as $t \to +\infty$, while the first member remains bounded. Q.E.D.

Proof of Theorem 2. Let $x(t)$ be a non-oscillatory solution of the equation (1), which is assumed to be positive on $[t_1, +\infty)$. The function $x^{(2n-1)}(t)$ is decreasing for $t \geqslant t_1$, because of our hypotheses. By integrating the equation (1) from t_1 to t, we have

$$-x^{(2n-1)}(t) + x^{(2n-1)}(t_1) = \int_{t_1}^{t} f(s)g(x, x', \cdots, x^{(2n-1)}) ds.$$

As it follows from Corollary in this section that $x^{(2n-1)}(t) \geqslant 0$ for $t \geqslant t_1$, we have

$$(2) \qquad x^{(2n-1)}(t_1) \geqslant \int_{t_1}^{t} f(s)g(x, x', \cdots, x^{(2n-1)}) ds \qquad \text{for } t \geqslant t_1.$$

By the proof of the lemma, x is increasing on the interval $[t_1, +\infty)$ and, by the corollary, we have the following two cases, which we shall take care of one by one.

Case 1:

$$x^{(m)}(\infty) = \begin{cases} 0 & \text{for } k+1 \leqslant m \leqslant 2n-1 \\ c & \text{for } m = k \\ +\infty & \text{for } 0 \leqslant m \leqslant k-1, \end{cases}$$

where $0 < k \leqslant 2n-1$ and $c \geqslant 0$. Then, our assumption implies that

$$\lim_{t\to+\infty} g(x, x', \cdots, x^{(2n-1)}) = G(k, c) > \varepsilon$$

for some positive ε. So there exists a $t_2 \geqslant t_1$ such that $g(x, x', \cdots, x^{(2n-1)}) > \varepsilon$ for all $t \geqslant t_2$. It is obvious that the inequality (2) remains valid if we replace t_1 by t_2. Thus we obtain

$$x^{(2n-1)}(t_2) \geqslant \varepsilon \int_{t_2}^{t} f(s)ds.$$

Clearly this leads to a contradiction.

Case 2:

$$x^{(m)}(\infty) = \begin{cases} 0 & \text{for } 1 \leqslant m \leqslant 2n-1 \\ c>0 & \text{for } m=0. \end{cases}$$

Then, given a positive number $\varepsilon < g(c, 0, \cdots, 0)$, there exists a $t_2 \geqslant t_1$ such that $g(c, 0, \cdots, 0) - \varepsilon < g(x, x', \cdots, x^{(2n-1)})$ for all $t \geqslant t_2$ and we find from (2)

$$x^{(2n-1)}(t_2) \geqslant \int_{t_2}^{t} f(s)g(x, x', \cdots, x^{(2n-1)})ds \geqslant [g(c, 0, \cdots, 0) - \varepsilon] \int_{t_2}^{t} f(s)ds$$

for all $t \geqslant t_2$, which again leads to a contradiction. Q.E.D.

3. **Remarks.** By considering the equation $x^{(3)} + x = 0$, we see that it may be impossible to replace $2n$ by $2n+1$ in Theorems 1 and 2.

As Kartsatos [2] has pointed out, we may improve the conditions in our theorems. For example, the homogeneity of g in Theorem 1 may be assumed only for positive λ and p may be any non-negative number. But we prefer here to the brevity of exposition at the expense of generality.

Recently, Bhatia [1] and Tomastik [3] also have shown oscillatory properties for some second-order differential equations. But their hypotheses are somewhat different from ours.

References

[1] N. B. Bhatia: An oscillation theorem. Notices Amer. Math. Soc., **13**, 243 (1966).

[2] A. G. Kartsatos: Some theorems on oscillations of certain non-linear second-order differential equations. Arch. der Math., **18**, 425-429 (1967).

[3] E. C. Tomastik: Oscillation of a non-linear second order differential equation. SIAM J. Appl. Math., **15**, 1275-1277 (1966).