24. On the Application of the Potential Theory to Martingales

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Introduction. In his recent book [1], P. A. Meyer mentioned a remark on the mapping of a set of processes into itself stated below, which enables one to apply the theory of potentials to that of martingales:

Let (Ω, F, P) be a probability space, and $\{F_n\}_{n \in N^+}$ be an increasing family of sub-σ-fields of F, where $N^+ = \{1, 2, 3, \cdots\}$. Put $S = \Omega \times N^+$, the product space, and attach to it the σ-field $\mathcal{F}$ consisting of the sets of the form $\bigcup_{n=1}^{\infty} A_n \times \{n\}$, where $A_n \in F_n$. Then any real-valued $\mathcal{F}$-measurable function may be identified with a process adapted to $\{F_n\}_{n \in N^+}$. If we denote by $\mathcal{A}$ the family of the sets of the form $\bigcup_{n=1}^{\infty} A_n \times \{n\}$, where $A_n \in F_n$ and $P(A_n) = 0$, then $\mathcal{A}$ is closed under countable union. We can define the mapping N of a certain class of processes into itself in the following manner:

$$(NX)_n = E(X_{n+1} \mid F_n).$$

N determines the process with the ambiguity of the values on the sets belonging to $\mathcal{A}$.

In this paper, we define a kernel which is a generalization of the usual kernel, establish the potential theory associated with the kernel, and deduce some theorems on martingales, though mostly already known, using above notions and the method suggested in Doob's paper [2].

§ 1. Sub-Markov pseudo kernels and potential theory. Let S be an abstract space and $\mathcal{F}$ be a σ-field of subsets of S. Let $\mathcal{A}$ be a subfamily of $\mathcal{F}$ closed under the operation of countable union. We denote by $\mathcal{P}^0$ the set of all $\mathcal{F}$-measurable functions on S with values in $[0, +\infty]$, and define the equivalence relation $\sim$ in $\mathcal{P}^0$ as follows:

$f \sim g$ if and only if $f(s) = g(s)$ on $S - A$ for some $A \in \mathcal{A}$.

We classify $\mathcal{P}^0$ by this equivalence relation and set $\mathcal{P} = \mathcal{P}^0/\sim$. Then we can naturally define the usual algebraic operations and limit processes in $\mathcal{P}$ from the corresponding operations in $\mathcal{P}^0$; this may be done by the same way as we do for function spaces on a measure space in case $\mathcal{A}$ is the totality of sets of measure zero.

Thus, in the sequel, we identify the element of $\mathscr{P}^0$ and the corresponding class in $\mathscr{P}$. We also do the same identification for the elements of $\mathscr{F}$, considering their indicator functions.

Definition 1. A map N of $\mathscr{P}$ into $\mathscr{P}$ is called a *sub-Markov pseudo kernel*, or simply a *kernel*, if the following (1) and (2) are satisfied:

(1) $N\left(\sum_{i=1}^{\infty} \lambda_i \cdot f_i\right) = \sum_{i=1}^{\infty} \lambda_i \cdot Nf$ for any $f \in \mathscr{P}$ and any non-negative constant λ_i;

(2) $N1 \leqq 1$, where 1 denotes the constant function with the value one.

Definition 2. $f \in \mathscr{P}$ is said to be excessive (resp. *invariant*) with respect to N if $f < +\infty$ and $Nf \leqq f$ (resp. $Nf = f$). If we omit the condition $f < +\infty$, f is said to be *excessive in the wider sense*.

Definition 3. The map $G = \sum_{n=0}^{\infty} N^n$ (where $N^0 = I$, identity) is called the *potential kernel associated with* N. For any $f \in \mathscr{P}$, Gf is called the *potential of* f.

Theorem 1. *The potential Gf of $f \in \mathscr{P}$ is excessive in the wider sense. If f is excessive and also if $N^\infty f = 0$, then f is the potential of $h = f - Nf \in \mathscr{P}$.*

The proof is easy.

Theorem 2. *If $f \in \mathscr{P}$ is excessive, then f has the unique Riesz decomposition of the form $f = g + h$, where g is invariant and h is excessive with $N^\infty h = 0$. This h can be written as a potential of some element of $\mathscr{P}$.*

To prove this it is sufficient to take $g = N^\infty f$ and $h = f - N^\infty f$.

Theorem 3. *Put $A = \{s; f(s) > 0\}$ for a given $f \in \mathscr{P}$. Then for any non-negative constant a and any $g \in \mathscr{P}$ which is excessive in the wider sense,*

$$a + g \geqq Gf \text{ on } A$$

implies

$$a + g \geqq Gf \text{ everywhere on } S.$$

The proof is the same as in Meyer [1].

Combining the above theorem with Theorem 1, we have the following corollary.

Corollary 1. *Suppose $f, g \in \mathscr{P}$, be excessive and $N^\infty f = 0$. If we put $A = \{s; f(s) > (Nf)(s)\}$, then*

$$a + g \geqq f \text{ on } A$$

implies

$a + g \geqq f$ *everywhere on* S, *where a is a non-negative constant.*

Definition 4. For $A \in \mathscr{F}$ we define a kernel I_A to be the multiplication of the indicator function of A. We set $N_A = N \cdot I_A$ and

$H_A = I_A + I_{A'} \cdot \left(\sum_{p=0}^{\infty} (N_{A'})^p \right) \cdot N_A$, where A' is the complementary set of A.

Theorem 4. *Let $f \in \mathscr{P}$ be excessive, and $A \in \mathscr{F}$. Then $f_A = H_A f$ is the smallest of all elements of $\mathscr{P}$ which are excessive and dominate f on the set A.*

The proof is the same as in Meyer [1].

Theorem 5. *Let a, b be two constant numbers such that $0 \leqq a < b$, and put $A = \{s; f(s,) \leqq a\}$ and $B = \{s; f(s) \geqq b\}$ for excessive $f \in \mathscr{P}$. Then we have the following three inequalities:*

(1) $1_B \leqq \dfrac{f}{b}$,

(2) $1_{AB} + 1_{ABAB} + 1_{ABABAB} + \cdots \leqq \dfrac{\min [f, b]}{b - a}$,

(3) $1_{BA} + 1_{BABA} + 1_{BABABA} + \cdots \leqq \dfrac{\min [f, a]}{b - a}$,

where 1 is the constant function with valuse one and 1_A should be understood in the same way as f_A in the preceeding theorem and $1_{AB} = (1_A)_B$ and so on.

The proof is the same as in Doob [2].

§ 2. Applications to the martingale theory. As one can easily see, the quartet $\{S, \mathscr{F}, \mathscr{A}, N\}$ defined in the introduction is an example of that defined in § 1 if we take for $\mathscr{P}^0$ the set of all non-negative processes adapted to $\{F_n\}_{n \in N^+}$. In this section we restrict ourselves to this example and all processes are supposed to be non-negative-valued. The theorems in this section correspond to those in § 1 with the same numbers.

Lemma 1. *$X \in \mathscr{P}$ is excessive (resp. invariant) if and only if it is a supermartingale (resp. martingale) taking finite values P-a.s. when it is considered as a process. The condition taking finite values is omitted if we replace excessive (resp. invariant) by excessive in the wider sense (resp. invariant in the wider sense).*

Theorem 1. *Let $\{X_n\}_{n \in N^+}$ be a finite valued supermartingale such that $\lim_{m \to \infty} E(X_m \mid F_n) = 0$ for any $n \in N^+$, then there exists a non-negative process $\{Y_n\}$ such that $X_n = \sum_{m=0}^{\infty} E(Y_{n+m} \mid F_n)$ for any $n \in N^+$.*

To prove this we have only to take $Y_n = X_n - E(X_{n+1} \mid F_n)$.

Theorem 2. *If $\{X_n\}$ is a finite-valued supermartingale, then $\{X_n\}$ has the unique Riesz decomposition of the form $X_n = Y_n + Z_n$, where $\{Y_n\}$ is a martingale and $\{Z_n\}$ is a supermartingale such that $\lim_{m \to \infty} E(Z_m \mid F_n) = 0$ for every $n \in N^+$.*

This may be shown by putting $Y_n = \lim_{m \to \infty} E(X_m \mid F_n)$ and $Z_n = X_n - Y_n$.

Theorem 3. *Let $\{X_n\}$ be a non-negative process and a be a non-negative constant, put $A_n = \{\omega; X_n(\omega) > 0\}$, and suppose $\{Y_n\}$ to*

be a supermartingale (*not necessarily finite-valued*). *Then*

$$a + Y_n \geqq \sum_{m=0}^{\infty} E(X_{n+m} \mid F_n) \text{ a.s. on } A_n \text{ for every } n \in N^+$$

implies

$$a + Y_n \geqq \sum_{m=0}^{\infty} E(X_{n+m} \mid F_n) \text{ a.s. on } \Omega \text{ for every } n \in N^+.$$

For this translation, it is enough to take $\bigcup_{n=1}^{\infty} A_n \times \{n\}$ for A in Theorem 3 of §1.

Corollary 1. *Suppose* $\{X_n\}, \{Y_n\}$ *be two supermartingales,* $\{X_n\}$ *being finite-valued and* $\lim_{m \to \infty} E(X_m \mid F_n) = 0$ *for any* $n \in N^+$. *If we put* $A_n = \{\omega; X_n > E(X_{n+1} \mid F_n)\}$. *Then, for any non-negative constant* a,

$$a + Y_n \geqq X_n \text{ a.s. on } A_n \text{ for every } n \in N^+$$

implies

$$a + Y_n \geqq X_n \text{ a.s. on } \Omega \text{ for every } n \in N^+.$$

Theorem 4. *Let* $\{X_n\}$ *be a supermartingale and* $\{A_n\}$ *be a family of sets such that* $A_n \in F_n$ *for every* n. *Then the class of supermartingales* $\{Z_n\}$ *such that* Z_n *dominates* X_n *on* A_n *for every* n *has the smallest element* $\{X_n^A\}$ *of the form*

$$X_n^A = E\left(\sum_{m=n}^{\infty} I_{nm} \cdot X_m \mid F_n\right) \quad \text{a.s.}$$

where I_{nn} *is the indicator function of* A_n *and* I_{nm} *is the indicator function of* $A_m \cap \left(\bigcup_{i=n}^{m-1} A_i\right)'$ *for* $m > n$.

Proof. We have only to translate $H_A \cdot f$ in Theorem 4 of §1 taking $\bigcup_{n=1}^{\infty} A_n \times \{n\}$ for A

$$X_n^A = I_{A_n} \cdot X_n + I_{A_n'} \cdot \sum_{m=n+1}^{\infty} E(I_{A_m} \cdot I_{A_{m-1}'} \cdots I_{A_{n+1}'} \cdot X_m \mid F_n)$$

$$= I_{A_n} \cdot X_n + \sum_{m=n+1}^{\infty} E(I_{A_m} \cdot I_{A_{m-1}'} \cdots I_{A_{n+1}'} \cdot I_{A_n'} \cdot X_m \mid F_n)$$

$$= I_{nn} \cdot X_n + \sum_{m=n+1}^{\infty} E(I_{nm} \cdot X_m \mid F_n)$$

$$= E\left(\sum_{m=n}^{\infty} I_{nm} \cdot X_m \mid F_n\right), \qquad \text{q.e.d.}$$

Lemma 2. *Let* $\{X_n\}$ *be a supermartingale. Given* $0 \leqq a \leqq b$, *we put* $A_n = \{\omega; X_n(\omega) \leqq a\}$, $B_n = \{\omega; X_n(\omega) \geqq b\}$. *Then*

  (1) $1_n^B = P(\{\omega; X_m \geqq b \text{ for some } m \geqq n\} \mid F_n)$ a.s.

  (2) $1_n^{AB} + 1_n^{ABAB} + 1_n^{ABABAB} + \cdots = E(D_n \mid F_n)$ a.s.

  (3) $1_n^{BA} + 1_n^{BABA} + 1_n^{BABABA} + \cdots = E(U_n \mid F_n)$ a.s.

where D_n *and* U_n *are respectively downcrossing and upcrossing numbers of the interval* $[a, b]$ *after the time* n, *and* 1 *is the constant process with the value one, which is a martingale, and* 1_n^A *is defined as in Theorem 4 and* $1_n^{AB} = (1^A)_n^B$ *and so on*.

Proof. Using Theorem 4 repeatedly we have, for example,

$1_n^{AB} = P(\{\omega; X_m(\omega) \text{ downcrosses } [a, b] \text{ at least once after } n\} \mid F_n)$.

By means of the relations of such type, this lemma can be proved easily.

Theorem 5. *Let $\{X_n\}$ be a supermartingale. Then using the same notation as in the preceding lemma, we have the following three inequalities,*

(1) $P(\{\omega;\ X_n(\omega) \geq b\ \text{for some}\ m \geq n\} \,|\, F_n) \leq \dfrac{X_n}{b}$ a.s.

(2) $E(D_n \,|\, F_n) \leq \dfrac{\min\,[X_n,\, b]}{b-a}$ a.s.

(3) $E(U_n \,|\, F_n) \leq \dfrac{\min\,[X_n,\, a]}{b-a}$ a.s.

(1) *reminds one of the well-known* Kolmogorov's *inequality and* (2) (3) *are modifications of the* Doob's *inequalities.*

These may be seen by combining Theorem 5 in §1 with the preceding lemma.

References

[1] P. A. Meyer: Probabilités et Potentiels. Hermann, Paris (1966).

[2] J. L. Doob: Proc. 4-th Berkeley Symp., **2**, 95-102 (1961).

[3] ——: Zeitsch. Wahr., **1**, 288-294 (1963).