56. A Note on the Automorphism Group of an Almost Complex Structure of Type (n, n')

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1. Introduction. The aim of this paper is to prove a theorem which asserts the finite-dimensionality of the automorphism group of a sub-elliptic almost complex structure of type (n, n') on a compact manifold. Its settling requires only a theorem of R. S. Palais and a theorem due to J. J. Kohn and L. Hörmander.

2. Definitions and theorems. Throughout this paper we assume the differentiability of class C^∞. Let M be a manifold of dimension $\imath n+n'$ and let S be a subbundle of its complexified tangent bundle whose fibers are of complex dimension n.

Definition 1. Let M, S be as above and $\bar{S}$ be the complex conjugate bundle of S. The pair (M, S) is called *an almost complex structure* (on M) *of type* (n, n') if it satisfies the following conditions; (i) S_p (the fiber of S over p) contains no real element except 0 (i.e. $S_p \cap \bar{S}_p = (0)$) for any $p \in M$ (ii) $[X, Y]$ is a cross-section of $S \oplus \bar{S}$ for any two cross-sections X, Y of S.

Definition 2. Let (M, S), (M', S') be two almost complex structures of type (n, n'). A diffeomorphism f of M onto M' is called *an isomorphism of* (M, S) *onto* (M', S') if $(df)_p$ maps S_p isomorphically onto $S'_{f(p)}$ for any $p \in M$. An isomorphism of (M, S) onto itself is called an *automorphism* of (M, S).

Let η be a real 1-form of M which vanishes on S and set $L_p^\eta(s, t) = i\langle (d\eta)_p | s \wedge \bar{t} \rangle$ for $s, t \in S_p$. L_p^η is a hermitian form on S_p. For any fixed $p \in M$, we denote the real vector space of all such L_p^η's by $\mathcal{L}_p$.

Definition 3. Let notations be as above. The almost complex structure (M, S) of type (n, n') is called *sub-elliptic* if it satisfies the following conditions; (i) $\dim_R \mathcal{L}_p = n'$ (ii) $\mathcal{L}_p$ contains no semi-definite form except 0 for any $p \in M$.

From now on we assume that M is compact. Let X^j $(j=1, 2, \cdots, \pi)$ be a series of cross-sections of S such that X_p^j $(j=1, 2, \cdots, \pi)$ span S_p for any $p \in M$, and let ξ^k $(k=1, 2, \cdots, \rho)$ be a series of (complex) forms such that each ξ^k vanishes on S and such that ξ_p^k, $\bar{\xi}_p^k$ $(k=1, 2, \cdots, \rho)$ span $T_p^*(\mathrm{M}) \otimes C$ for any $p \in M$. (The existence of such ξ^k's is guaranteed by the condition (i) of Definition 1.) Assuming that the Sobolev norms $\| \ \|_{(s)}$ (s: real) on $C^\infty(M)$ are already introduced as usual, we also introduce $\| \ \|_{(s)}$ for vector fields by setting:

$$\| X \|_{(s)} = \sum_{k=1}^{\rho} (\| \xi^k(X) \|_{(s)} + \| \bar{\xi}^k(X) \|_{(s)}) \quad X: \text{a vector field.}$$

Now we collect a few results which will be used in the proof of our theorem.

Theorem 1. *Let M, S, X^j $(j=1, 2, \cdots, \pi)$ be as above and define a differential operator $\mathfrak{X}$: $C^\infty(M) \to (C^\infty(M))^\pi$ by setting $\mathfrak{X}u = (X^1u, \cdots, X^\pi u)$. Then $\mathfrak{X}$ is sub-elliptic* (i.e. $\exists c > 0$ $\forall u \in C^\infty(M)$ $\| u \|_{(\frac{1}{2})} \leq C \| \mathfrak{X}u \|_{(0)}$) *if and only if (M, S) is sub-elliptic.*

This theorem was first proved by J. J. Kohn [1] in the case $n'=1$. The general case is an easy consequence of Hörmander [2]. (See Theorems 1.1.5, Theorems 1.2.3.)

Now let f_t $(t \in R)$ be a 1-parameter subgroup of automorphisms of (M, S) and Y be its generating vector field. Then $[X, Y]$ is a cross-section of S for any cross-section X of S. The converse is also true since M is compact, and we denote the Lie algebra of all such Y's by $\mathfrak{A}(M, S)$ (i. e. $\mathfrak{A}(M, S) = [Y \in \Gamma(T(M)) \mid [X, Y] \in \Gamma(S)$ for any $X \in \Gamma(S)]$). Then a theorem of R. S. Palais [3] asserts.

Proposition 1. *The automorphism group of (M, S) is a Lie transformation group if and only if $\mathfrak{A}(M, S)$ is finite-dimensional.*

We are now ready to prove the theorem announced in the introduction.

Theorem 2. *The automorphism group of (M, S) is a Lie transformation group on M if (M, S) is a sub-elliptic almost complex structure of type (n, n') on a compact manifold M.*

Proof. By Proposition 1 it is sufficient to prove that $\mathfrak{A}(M, S)$ is finite-dimensional. Suppose that Y is in $\mathfrak{A}(M, S)$ and that $\xi^k (k=1, 2, \cdots, \rho)$, $X^j (j=1, 2, \cdots, \pi)$ be as before. Taking the Lie derivative of $\xi^k(X^j)=0$ with respect to Y, we have

$$\mathcal{L}_Y(\xi^k)(X^j) + \xi^k([X^j, Y]) = 0.$$

Since the second term vanishes by the definition of $\mathfrak{A}(M, S)$, we obtain

$$\mathcal{L}_Y(\xi^k)(X^j) = 0.$$

Using the formula $\mathcal{L}_Y(\omega) = d(Y \lrcorner\, \omega) + Y \lrcorner\, d\omega$, we rewrite this into

$$X^j(\xi^k(Y)) = \langle d\xi^k \mid X^j \wedge Y \rangle.$$

Notice that the right hand side contains no differentiation of Y. Thus, applying Theorem 1 to the above

$$\| \xi^k(Y) \|_{(\frac{1}{2})} \leq C \| Y \|_{(0)}$$

where C is a positive constant independent of $Y \in \mathfrak{A}(M, S)$. Since Y is a real vector field, we obtain also

$$\| \bar{\xi}^k(Y) \|_{(\frac{1}{2})} = \| \overline{\xi^k(Y)} \|_{(\frac{1}{2})} = \| \xi^k(Y) \|_{(\frac{1}{2})} \leq C \| Y \|_{(0)}.$$

Then by the definition of the $\| \ \|_{(s)}$ for vector fields, we get

$$\| Y \|_{(\frac{1}{2})} \leq C \| Y \|_{(0)} \qquad \text{for} \quad Y \in \mathfrak{A}(M, S)$$

if we take some larger $C>0$. Thus $\mathfrak{A}(M, S)$ is finite-dimensional.

Q.E.D.

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References

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