51. On a New Positive Linear Polynomial Operator

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(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1968)

In this note we introduce a new positive linear polynomial operator: $P_m^{[\alpha]}(f;x)=P_m^{[\alpha]}(f(t);x)$, corresponding to a function $f=f(x)$, defined on the interval $[0,1]$, and to a parameter $\alpha\geqq 0$, which may depend only on the natural number m. This operator is

$$(1)\qquad P_m^{[\alpha]}(f;x)=\sum_{k=0}^{m} w_{m,k}(x;\alpha)f\left(\frac{k}{m}\right),$$

where

$$w_{m,k}(x;\alpha) =\binom{m}{k}\frac{x(x+\alpha)\cdots(x+\overline{k-1}\alpha)(1-x)(1-x+\alpha)\cdots(1-x+\overline{m-k-1}\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+\overline{m-1}\alpha)}.$$

One observes that it represents a *polynomial* of *degree* m.

Because $\alpha\geqq 0$, we have $w_{m,k}(x;\alpha)\geqq 0$ for $x\in[0,1]$. Therefore the linear (additive and homogeneous) operator (1) is *positive* on the interval $[0,1]$. In fact we have here a class of operators depending on the parameter α.

First we should remark that for $\alpha=0$ the operator (1) reduces to the Bernstein polynomial

$$(2)\qquad B_m(f;x)=\sum_{k=0}^{m} p_{m,k}(x)f\left(\frac{k}{m}\right),\quad p_{m,k}(x)=\binom{m}{k}x^k(1-x)^{m-k}.$$

Then we wish to make the remark that if we choose $\alpha=0(m^{-1})$ and use the change of variable $x=\frac{n}{m}y$, n being a natural number not depending on m, then—denoting again the variable by x—we obtain from our operator the Mirakyan operator

$$(3)\qquad M_n(f;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right).$$

After these preliminaries we can state several theorems, the proofs of which will appear in the journal: Rev. Roumaine Math. Pures Appl.

Theorem 1. *If the parameter α has a fixed non-negative value in each term of the sequence $\{P_m^{[\alpha]}(f;x)\}$, then there exists the following relationship*

$$P_{m+1}^{[\alpha]}(f;x)-P_m^{[\alpha]}(f;x)$$

$$(4)\qquad =-\frac{1}{m(m+1)}\sum_{\nu=0}^{m-1}\frac{(x+\nu\alpha)(1-x+\overline{m-\nu-1}\alpha)}{(1+m\alpha)(1+\overline{m-1}\alpha)}w_{m-1,\nu}(x;\alpha)\times\left[\frac{\nu}{m},\frac{\nu+1}{m+1},\frac{\nu+1}{m};f\right],$$

where $\left[\frac{\nu}{m},\frac{\nu+1}{m+1},\frac{\nu+1}{m};f\right]$ *represents the divided difference of* f *on the indicated nodes.*

From this theorem one deduces the following

Corollary. *Let* $\alpha\geqq 0$ *has a fixed value. If* f *is convex of first order on the interval* $[0,1]$, *then the sequence of polynomials* $\{P_m^{[\alpha]}(f;x)\}$ *is decreasing on the interval* $(0,1)$. *If* f *is non-concave of first order on* $[0,1]$ *then the sequence* $\{P_m^{[\alpha]}(f;x)\}$ *is non-increasing on* $[0,1]$.

Selecting $\alpha=0$ we are led to the known monotonicity properties of the sequence of Bernstein's polynomials, studied first by Temple [9].

If we take into account that one can deduce from (4), as a limiting case, the following equality,

$$M_{n+1}(f;x)-M_n(f;x)=-\frac{x}{n(n+1)}\sum_{\nu=0}^{\infty}\frac{(nx)^\nu}{\nu!}e^{-nx}\left[\frac{\nu}{n},\frac{\nu+1}{n+1},\frac{\nu+1}{n};f\right],$$

it will be easy to state the corresponding monotonicity properties of the sequence of Mirakyan's operators, which have been investigated directly first by Cheney and Sharma [2].

We obtained for the operator (1) an expression in terms of finite differences of f.

Theorem 2. *The operator* (1) *can be represented in the following form*

$$(5)\qquad P_m^{[\alpha]}(f;x)=f(0)+\sum_{j=1}^{m}\binom{m}{j}\frac{x(x+\alpha)\cdots(x+\overline{j-1}\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+\overline{j-1}\alpha)}\Delta_{1/m}^{j}f(0),$$

where $\Delta_{1/m}^{j}f(0)$ *is the finite difference of order* j, *with the step* $1/m$ *and the starting point* 0, *of the function* f.

The key role in proving this theorem is played by the following

Lemma. *If* $x\in(0,1)$ *and* $\alpha>0$, *then we can write*

$$w_{m,k}(x;\alpha)=\binom{m}{k}\frac{B\left(\frac{x}{\alpha}+k,\frac{1-x}{\alpha}+m-k\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)},$$

where $B(a,b)$ *represents the beta function.*

By using the representation (5) of the operator (1) we deduce at once the following identities

$$P_m^{[\alpha]}(1\,;x)=1,\quad P_m^{[\alpha]}(t\,;x)=x,$$
$$P_m^{[\alpha]}(t^2\,;x)=\frac{1}{1+\alpha}\left[\frac{x(1-x)}{m}+x(x+\alpha)\right]. \tag{6}$$

We now state the main theorem of this note.

Theorem 3. *If $f\in C[0,1]$ and $0\leqq\alpha=\alpha(m)\to 0$ as $m\to\infty$, then the sequence $\{P_m^{[\alpha]}(f\,;x)\}$ converges to $f(x)$ uniformly on $[0,1]$.*

In order to prove this theorem it suffices to refer to a well known theorem of Bohman-Korovkin, (see, e.g., [1] or [4]) and to make use of the identities (6).

The following theorem enables us to see the order of approximation of a continuous function f by our operator (1).

Theorem 4. *Let denote by $\omega(\delta)=\omega(f\,;\delta)$ the modulus of continuity of f. If $f\in C[0,1]$ and $\alpha\geqq 0$, then we have*

$$|f(x)-P_m^{[\alpha]}(f\,;x)|\leqq\frac{3}{2}\omega\left(\sqrt{\frac{1+\alpha m}{m+\alpha m}}\right). \tag{7}$$

The main steps in the proof of this theorem consist in proving the following two inequalities

$$|f(x)-P_m^{[\alpha]}(f\,;x)|\leqq\left(1+\frac{1}{\delta}\sum_{k=0}^{m}w_{m,k}(x\,;\alpha)\,|x-\frac{k}{m}|\right)\omega(\delta)\quad(\delta>0),$$
$$\sum_{k=0}^{m}w_{m,k}(x\,;\alpha)\left|x-\frac{k}{m}\right|\leqq\frac{1}{2}\sqrt{\frac{1+\alpha m}{m+\alpha m}}.$$

We should remark that for $\alpha=0$ the inequality (7) reduces to the known inequality of Popoviciu [6]. The corresponding inequality for the operator (3) was recently given by Müller [5].

If one further assumes that f possesses a continuous derivative on $[0,1]$, then we may state

Theorem 5. *If $f\in C^1[0,1]$, then we have the inequality*

$$|f(x)-P_m^{[\alpha]}(f\,;x)|\leqq\frac{3}{4}\sqrt{\frac{1+\alpha m}{m+\alpha m}}\,\omega_1\left(\sqrt{\frac{1+\alpha m}{m+\alpha m}}\right),$$

$\omega_1(\delta)$ being the modulus of continuity of f'.

If $\alpha=0$ then it reduces to an inequality of Lorentz [3].

It is readily seen that in the case of Mirakyan's operator we have

$$|f(x)-M_n(f\,;x)|\leqq(a+\sqrt{a})\frac{1}{\sqrt{n}}\omega_1\left(\frac{1}{\sqrt{n}}\right),$$

where $a>0$, $x\in[0,a]$ and $f\in C^1[0,a]$.

The remainder term of the approximation formula

$$f(x)=P_m^{[\alpha]}(f\,;x)+R_m^{[\alpha]}(f,x)$$

can be expressed by means of divided differences.

Theorem 6. *We have the equality*

$$R_m^{[\alpha]}(f;x)=\frac{x-1}{m(1+\overline{m-1}\alpha)}\sum_{k=0}^{m-1}(x+\alpha k)w_{m-1,k}(x;\alpha)\left[x,\frac{k}{m},\frac{k+1}{m};f\right]$$
$$+\frac{\alpha}{1+\overline{m-1}\alpha}\sum_{k=0}^{m-1}(\overline{m-1}x-k)w_{m-1,k}(x;\alpha)\left[x,\frac{k}{m};f\right].$$

If $\alpha=0$ then it reduces immediately to

$$R_m(f;x)=-\frac{x(1-x)}{m}\sum_{k=0}^{m-1}p_{m-1,k}(x)\left[x,\frac{k}{m},\frac{k+1}{m};f\right],$$

which corresponds to the approximation of f by the Bernstein polynomial (2). It was first obtained by us in [7].

Finally, we give an asymptotic estimate of the remainder $R_m^{[\alpha]}(f;x)$.

Theorem 7. *Let $\alpha=\alpha(m)\to 0$ as $m\to\infty$. If f is bounded on* [0, 1] *and possesses a second derivative at a point $\bar{x}$ of* [0, 1], *then*

$$R_m^{[\alpha]}(f;\bar{x})=-\frac{1+\alpha m}{1+\alpha}\cdot\frac{\bar{x}(1-\bar{x})}{2m}f''(\bar{x})+\frac{\varepsilon_m^{[\alpha]}(\bar{x})}{m},$$

where $\varepsilon_m^{[\alpha]}(\bar{x})$ tends to 0 *when m tends to ∞.*

This theorem represents an extension to our operator (1) of a theorem due to Voronovskaja (see, e.g., [3]) in the special case of Bernstein's polynomial (2).

References

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