45. On Potential Kernels Satisfying the Complete Maximum Principle

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Let $(E, \mathcal{E})$ be a measurable space and V a proper kernel on $(E, \mathcal{E})$ which satisfies the complete maximum principle. It is known that if $V1$ is bounded, there then exists a sub-Markov resolvent $(V_p)_{p>0}$ such that

$$V = \lim_{p \to 0} V_p \tag{1}$$

(see [4, p. 206]). On the other hand, if $V1$ is unbounded, there is such a kernel V for which the condition (1) is never satisfied by any sub-Markov resolvent $(V_p)_{p>0}$ (for an example, see also [4, p. 206]).

In this note we shall give a *sufficient* condition under which the kernel V can be expressed in the form (1) by a sub-Markov resolvent $(V_p)_{p>0}$. The condition is stated in terms of the pseudo-réduite and it is similar to that of Theorem 7 of Meyer [5].*) Our result contains Theorem II of Lion [3] as a special case.

1. Preliminary results. Throughout this note notations and terminology are taken from Meyer [4]. We will omit the definitions of a *proper* [resp. *sub-Markov*] kernel, a *sub-Markov resolvent* (we shall call it simply a *resolvent*) and a *supermedian function* with respect to a resolvent. A subset of E and a function on E are always assumed to be $\mathcal{E}$-measurable, so we will omit the phrase "$\mathcal{E}$-measurable".

Let A be a subset of E and h a supermedian function with respect to a resolvent $(V_p)_{p>0}$. Then the collection of supermedian functions that dominate h on A has the smallest element, which will be called the *pseudo-réduite* of h on A and denoted by $H_A h$ [4, p. 200]. A resolvent $(V_p)_{p>0}$ is said to be *closed* if the kernel V_0 defined by $V_0 = \lim_{p \to 0} V_p$ is proper. If $(V_p)_{p>0}$ is closed and $V_0 f$ $(f \geqq 0)$ is finite, then the function $V_0 f$ is supermedian with respect to $(V_p)_{p>0}$. If the support of f is contained in A, then $H_A V_0 f = V_0 f$ [5, p. 231].

Let U be any proper kernel on $(E, \mathcal{E})$. A non-negative function

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*) Meyer discussed the following problem and gave a necessary and sufficient condition for the kernel U. "When is the proper kernel U generated by a sub-Markov kernel P in the sense $U = \sum_{n=0}^{\infty} P^n$". This is closely connected to our problem.

h on E is said to be *U-quasi-excessive* if, whenever $U|f|$ is bounded, the relation $h \geqq Uf$ on the set $\{f>0\}$ implies $h \geqq Uf$ everywhere. If $(V_p)_{p>0}$ is closed, then a non-negative function h is supermedian with respect to $(V_p)_{p>0}$ when and only when it is V_0-quasi-excessive [5, p. 230].

2. The complete maximum principle. We shall say that a proper kernel V on $(E, \mathcal{E})$ satisfies the *complete maximum principle* if it has the following property:

(C. M. P.) If a constant $a \geqq 0$ and if $V|f|$ is finite, the relation $Vf \leqq a$ on the set $\{f>0\}$ implies $Vf \leqq a$ everywhere.

Let V a proper kernel satisfying (C. M. P.) and u, a function such that $u(x)>0$ for all $x \in E$ and Vu is bounded (such a function always exists since V is proper). Then the kernel $\tilde{V}$ defined by $\tilde{V}(x, A) = \int_A V(x, dy)u(y)$ satisfies also (C. M. P.). Since $\tilde{V}1$ is bounded, there exists a closed resolvent $(\tilde{V}_p)_{p>0}$ such that $\tilde{V}=\tilde{V}_0$. A function h is V-quasi-excessive if and only if it is $\tilde{V}$-quasi-excessive. Therefore, for any V-quasi-excessive function h and any subset A, we can define the pseudo-réduite $H_A h$. From (C. M. P.) it follows that a potential Vf of non-negative function f is V-quasi-excessive, so that the pseudo-réduite $H_A Vf$ is well defined. Put $G^p = I + pV$ for each $p>0$.

Lemma 1. *If h is V-quasi-excessive and if $G^p|f|$ is finite, then the relation $G^p f \leqq h$ on the set $\{f>0\}$ implies $G^p f \leqq h - f^-$ everywhere, where $f^- = \sup(0, -f)$.*

Proof. On the set $\{f>0\}$, we have $pVf \leqq G^p f \leqq h$. However, since h is V-quasi-excessive, $pVf \leqq h$ everywhere. Hence $pVf - f^- \leqq h - f^-$ everywhere, which implies $G^p f \leqq h - f^-$ on the set $\{f \leqq 0\}$. Thus $G^p f \leqq h - f^-$ everywhere.

Corollary 1. *Any V-quasi-excessive function is G^p-quasi-excessive.*

Corollary 2. *G^p satisfies the reinforced maximum principle as follows:*

(R. M. P.) *If a constant $a \geqq 0$ and if $G^p|f|$ is finite, the relation $G^p f \leqq a$ on the set $\{f>0\}$ implies $G^p f \leqq a - f^-$ everywhere.*

Since condition (R. M. P) implies condition (C. M. P), for any G^p-quasi-excessive function h and a subset A, we can define the pseudo-réduite $H_A^p h$ of h on A with respect to G^p. If h is V-quasi-excessive, then $H_A^p h \leqq H_A h$, for $H_A h$ is a G^p-quasi-excessive function that dominates h on A. From (R. M. P.) it follows that if $0 \leqq G^p f \leqq 1$, then $0 \leqq G^p f - f \leqq 1$. Therefore, if $G^p f = 0$, then $f=0$ and if $f \geqq 0$, then $G^p f \geqq f$. Condition (R. M. P.) is equivalent to condition (R. M.) of Meyer [5] and hence, for any bounded G^p-quasi-excessive function h, we can find a sequence of non-negative functions $(g_n)_{n \geqq 1}$ such that

$G^p g_n$ increases to h as $n\to\infty$ [5, p. 235].

3. Construction of the resolvent. Let V be a proper kernel satisfying (C. M. P.). Let $\boldsymbol{B}$ be the Banach space of all real valued, bounded functions with the uniform norm and $\boldsymbol{B}_0$, the collection of f such that both f and Vf are in $\boldsymbol{B}$. Further $\boldsymbol{B}^+$ [resp. $\boldsymbol{B}_0^+$] denotes the cone of all non-negative functions of $\boldsymbol{B}$ [resp. $\boldsymbol{B}_0$]. In this section we assume that V satisfies the following additional condition:

(N) For any function $f\in \boldsymbol{B}_0^+$ and any increasing sequence of subsets $(A_n)_{n\geqq 1}$ with $\cup_{n\geqq 1}A_n=E$,
$$\lim_{n\to\infty} H_{E\setminus A_n}Vf=0.$$

The next lemma is a slight modification of Lemma 9 of Meyer [5].

Lemma 2. *If a sequence of non-negative functions $(g_n)_{n\geqq 1}$ converges to a function g and if there is a function $f\in \boldsymbol{B}_0^+$ such that $G^p g_n\leqq Vf$ for all $n\geqq 1$, then $G^p g_n$ converges to $G^p g$ as $n\to\infty$.*

Proof. Take a function v in $\boldsymbol{B}_0^+$, positive everywhere, and put $A_m=\{Vf\leqq mv\}$ for each positive integer m. Then the sequence $(A_m)_{m\geqq 1}$ is increasing to E. Since $g_n\leqq G^p g_n\leqq Vf$, we have $\{g_n>mv\}\subseteqq B_m=E\setminus A_m$. Hence, for all m, $n\geqq 1$,
$$\int_{\{g_n>mv\}} G^p(x,dy)g_n(y)\leqq \int_{B_m} G^p(x,dy)g_n(y)$$
$$=G^p(I_{B_m}g_n)(x)=H^p_{B_m}G^p(I_{B_m}g_n)(x)\leqq H^p_{B_m}Vf(x)\leqq H_{B_m}Vf,$$
where I_A denotes the indicator of a set A. Since $H_{B_m}Vf$ converges to 0 when $m\to\infty$, the sequence of non-negative functions $(g_n/v)_{n\geqq 1}$ is uniformly integrable with respect to each bounded measure $G^p(x,dy)v(y)$. Therefore, $G^p g_n$ converges to $G^p g$ when $n\to\infty$.

Lemma 3. *There is a family of mappings $(V_p)_{p>0}$ from $\boldsymbol{B}_0^+$ to $\boldsymbol{B}_0^+$ such that* (a) $(I+pV)V_pf=Vf$, (b) *if $f\leqq 1$, then $pV_pf\leqq 1$,* (c) $V_p(af+bg)=aV_pf+bV_pg$, *where a and b are non-negative constants, and* (d) $V_pf-V_qf+(p-q)V_pV_qf=0$.

Proof. Let $f\in \boldsymbol{B}_0^+$. Noting that Vf is G^p-quasi-excessive, choose a sequence of non-negative functions $(g_n)_{n\geqq 1}$ such that $G^p g_n$ increases to Vf when $n\to\infty$. Since the sequence $G^p g_n-g_n$ is increasing and $G^p g_n-g_n\leqq Vf$, $g_n=G^p g_n-(G^p g_n-g_n)$ converges to a function g as $n\to\infty$. Define $V_pf=g$. By Lemma 2, we have $G^pV_pf=Vf$, proving $V_pf\in \boldsymbol{B}_0^+$ and (a). We should note that V_pf is independent of the choice of $(g_n)_{n\geqq 1}$, because $I+pV$ satisfies (R. M. P.). Next, let $f\leqq 1$, then
$$1\geqq f=(I+pV)f-pVf=(I+pV)(f-pV_pf).$$
Noting that $I+pV$ satisfies (R. M. P.), we have
$$1\geqq (I+pV)(f-pV_pf)-(f-pV_pf)=pV_pf,$$
proving (b). Assertion (c) is evident. Finally, let p, $q>0$, and $f\in \boldsymbol{B}_0^+$, then

$$(I+pV)(V_pf-V_qf)=(I+pV)V_pf-(I+qV)V_qf+(p-q)VV_qf$$
$$=(p-q)VV_qf=(I+pV)((q-p)V_pV_qf).$$

Thus, using (R. M. P.) again, we have $V_pf-V_qf=(q-p)V_pV_qf$. Therefore the lemma was proved.

For each $f\in \boldsymbol{B}_0$, define $V_pf=V_pf^+-V_pf^-$, where $f^+=\sup(0,f)$. Noting that $V_pf\leqq Vf$ for all $f\in \boldsymbol{B}_0^+$ and that V is a kernel on $(E,\mathcal{E})$, we can easily verify that, for each $x\in E$, the linear functional: $f\to V_pf(x)$ defined on $\boldsymbol{B}_0$ has all the properties of Daniell integral. Therefore there is a measure $V_p(x,\ \ )$ on E for which any function f in $\boldsymbol{B}_0$ is measurable and

$$V_pf(x)=\int_E V_p(x,dy)f(y).$$

Since any function in $\boldsymbol{B}^+$ is obtained as the limit of an increasing sequence of functions in $\boldsymbol{B}_0^+$, $V_p(x,\ \ )$ is a measure on $(E,\mathcal{E})$. Then we may consider $(V_p)_{p>0}$ as a sub-Markov resolvent.

Theorem. *Let V be a proper kernel which satisfies the complete maximum principle. Under condition (N), there is a closed sub-Markov resolvent* $(V_p)_{p>0}$ *such that* $V=V_0$. *Such a resolvent is unique.*

Proof. The uniqueness of such a resolvent is proved in [4, p. 205]. Let $(V_p)_{p>0}$ be a resolvent constructed above. $V_pf\leqq Vf$ for all $f\in \boldsymbol{B}_0^+$ and $p>0$, then $V_0f\leqq Vf$ for all $f\in \boldsymbol{B}_0^+$, so that the resolvent is closed. So we have only to prove $V=V_0$. For this purpose it is sufficient that we prove $Vf=V_0f$ for all $f\in \boldsymbol{B}_0^+$. Let $f\in \boldsymbol{B}_0^+$ and $p>0$. From (a) of Lemma 3, we have $(I+pV)pV_pf=pVf$, $(I+pV)(pV_p)^2f=pV(pV_p)f,\cdots,(I+pV)(pV_p)^{n+1}f=pV(pV_p)^nf,\cdots$. Therefore

$$\sum_{k=1}^{n}(pV_p)^kf+pV(pV_p)^nf=pVf \quad \text{for all} \quad n\geqq 1.$$

Hence, $V(pV_p)^nf\leqq Vf$ and $\lim_n(pV_p)^nf=0$, which implies $\lim_n V(pV_p)^nf=0$ by Lemma 2. Therefore

$$\sum_{k=1}^{\infty}(pV_p)^kf=pVf. \tag{2}$$

On the other hand, since the resolvent $(V_p)_{p>0}$ is closed, the left hand side of (2) is equal to pV_0f [4, p. 193] and so $Vf=V_0f$. Thus the theorem is proved.

References

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