107. σ-Spaces and Closed Mappings. II

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1. This is the continuation of our previous paper [6] in which we proved the following:

Theorem. *Let X be a normal T_1 σ-space and f a closed mapping*[1] *of X onto a topological space Y. Then Y is a normal T_1 σ-space such that the set $\{y \mid \partial f^{-1}(y)$ is not countably compact$\}$ is σ-discrete in Y, where $\partial f^{-1}(y)$ denotes the boundary of $f^{-1}(y)$.*

The purpose of this paper is to consider some applications of the above theorem to σ_0 spaces and to prove three theorems below. We shall say that a topological space X is *countable-dimensional* or σ_0 if it is the sum of X_i, $i=1, 2, \cdots$, with $\dim X_i \leqq 0$, where $\dim X_i$ denotes the covering dimension of X_i defined by means of finite open coverings, and that X is *uncountable-dimensional* if it is not σ_0.

Theorem 1. *Let X be a collectionwise normal T_1 σ-space and f a closed mapping of X onto a topological space Y such that $\partial f^{-1}(y)$ is countable for each $y \in Y$ or discrete for each $y \in Y$. Then Y is a countable sum of subspaces, each of which is homeomorphic to a subspace of X.*

Theorem 2. *Let X be a collectionwise normal σ_0 and T_1 σ-space and f a closed mapping of X onto an uncountable-dimensional space Y. Then Y contains an uncountable-dimensional subset N of Y such that $\partial f^{-1}(y)$ is uncountable for each $y \in Y$.*

Theorem 3. *Let X be a collectionwise normal σ_0 and T_1 σ-space and f a closed mapping of X onto an uncountable-dimensional space Y. Then Y contains an uncountable-dimensional subset Y such that $\partial f^{-1}(y)$ is dense-in-itself, non-empty and compact for each $y \in Y$.*

The first two theorems are generalizations of the results obtained by A. Arhangel'skii [2] which were proved in the case of spaces with countable nets and the last one is a generalization of K. Nagami's theorem [4] which was proved in the case of metric space, all of which concerned with a problem of P. Alexandroff [1] on the effect of closed mappings on countable-dimensional spaces.

2. To prove our results we need a few preliminaries.

Lemma 1. *Let $\mathfrak{F}$ be a collection of subsets of a topological space*

1) All mappings in this paper are *continuous*.

X and $\mathfrak{H}$ a σ-discrete collection of subsets of X such that each element of $\mathfrak{H}$ intersects with at most countable number of elements of $\mathfrak{F}$. Then $\mathfrak{F}\wedge\mathfrak{H}=\{F\cap H \mid F\in\mathfrak{F}, H\in\mathfrak{H}\}$ is σ-discrete in X.

Proof. Let $\mathfrak{F}=\{F_\alpha \mid \alpha\in\mathfrak{A}\}$ be the given collection and $\mathfrak{H}=\bigcup_{i=1}^{\infty}\mathfrak{H}_i$, $\mathfrak{H}_i=\{H_\lambda \mid \lambda\in\Lambda_i\}$ the given discrete collection for $i=1,2,\cdots$, and let

$$\mathfrak{A}_\lambda=\{\alpha \mid \alpha\in\mathfrak{A}, F_\alpha\cap H_\lambda\neq\phi\}=\{\alpha_1^\lambda, \alpha_2^\lambda, \cdots\}$$

for each $\lambda\in\bigcup_{i=1}^{\infty}\Lambda_i$. Besides, let us put

$$K_{\alpha\lambda}=F_\alpha\cap H_\lambda \quad \text{for each} \quad \alpha\in\mathfrak{A} \quad \text{and} \quad \lambda\in\bigcup_{i=1}^{\infty}\Lambda_i,$$

and

$$\mathfrak{K}_{jk}=\{K_{\alpha\lambda} \mid \alpha=\alpha_j^\lambda, \lambda\in\Lambda_k\} \quad \text{for} \quad j,k=1,2,\cdots.$$

Then it is easily seen that $\mathfrak{K}_{jk}$ is discrete in X for each j,k, and $\mathfrak{F}\wedge\mathfrak{H}=\bigcup_{j,k=1}^{\infty}\mathfrak{K}_{jk}$. This completes the proof.

Proposition. *For a collectionwise normal T_1 space X the following properties are equivalent:*

(i) *X has a σ-locally finite net,*

(ii) *X has a σ-discrete net.*

Proof. Since it is clearly (ii)→(i), we prove (i)→(ii), only. Let $\mathfrak{B}=\bigcup_{n=1}^{\infty}\mathfrak{B}_n$ be a σ-locally finite net for X. Since a collectionwise normal T_1 space with a σ-locally finite net is paracompact (cf. [5]), there exists a σ-discrete (open) covering $\mathfrak{H}$ of X such that each element of $\mathfrak{H}$ intersects with at most finite number of elements of $\mathfrak{B}_n$ for $n=1,2,\cdots$ (cf. [7]). By Lemma 1 $\mathfrak{C}_n=\mathfrak{B}_n\wedge\mathfrak{H}_n$ is a σ-discrete collection in X for $n=1,2,\cdots$, and $\mathfrak{C}=\bigcup_{n=1}^{\infty}\mathfrak{C}_n$ is also σ-discrete in X. Since $\mathfrak{B}$ is a net for X, $\mathfrak{C}$ is a net for X, too. This completes our proof.

Now we shall prove the following two lemmas in an analogous way as the case of metric spaces by K. Nagami [4].

Lemma 2. *Let X be a collectionwise normal T_1 space and $\mathfrak{B}$ a σ-locally finite net for X such that each $B\in\mathfrak{B}$ is a σ_0 space. Then X is a σ_0 space.*

Proof. By Proposition we can assume that $\mathfrak{B}=\bigcup_{n=1}^{\infty}\mathfrak{B}_n$ is a σ-discrete net for X such that each $B\in\mathfrak{B}$ is σ_0. Hence $B_n^*=\bigcup\{B \mid B\in\mathfrak{B}_n\}$ is also σ_0 and $X=\bigcup_{n=1}^{\infty}B_n^*$ is σ_0, too.

Lemma 3. *Let X and Y be collectionwise normal T_1 σ-spaces and f a closed mapping of X onto Y such that $f^{-1}(y)$ is compact and is not dense-in-itself for each $y\in Y$. If X is σ_0, then Y is σ_0, too.*

Proof. Let $\mathfrak{B}=\bigcup_{n=1}^{\infty}\mathfrak{B}_n$ be a σ-discrete net for X (see Proposition) where $\mathfrak{B}_n=\{B_\alpha \mid \alpha \in \mathfrak{A}_n\}$ for $n=1,2,\cdots$. Since X is regular, we can assume that each $B \in \mathfrak{B}$ is closed in X. Since X is σ_0, each $B \in \mathfrak{B}$ is σ_0. By the assumption $f^{-1}(y)$ contains an isolated point $x(y)$. Since $\mathfrak{B}$ is a net for X, there exist an n and an $\alpha(y)$ of $\bigcup_{n=1}^\infty \mathfrak{A}_n$ such that $B_{\alpha(y)} \cap f^{-1}(y) =\{x(y)\}$. Let

$$Y_\alpha=\{y \mid \alpha(y)=\alpha\} \quad \text{for each} \quad \alpha \in \bigcup_{n=1}^\infty \mathfrak{A}_n$$

and

$$Y_n=\bigcup\{Y_\alpha \mid \alpha \in \mathfrak{A}_n\} \quad \text{for} \quad n=1,2,\cdots.$$

Then $Y=\bigcup_{n=1}^\infty Y_n$. Let

$$X_\alpha=\{x(y) \mid y \in Y_\alpha\} \quad \text{for each} \quad \alpha \in \bigcup_{n=1}^\infty \mathfrak{A}_n$$

and

$$X_n=\{x(y) \mid y \in Y_n\} \quad \text{for each} \quad n=1,2,\cdots.$$

Then $f(X_\alpha)=Y_\alpha$ and $f(X_n)=Y_n$. Since $f \mid B_\alpha$ is closed and $B_\alpha \cap f^{-1}(Y_\alpha) =X_\alpha$, f maps X_α onto Y_α homeomorphically. Since X is perfectly normal (cf. [5]) and σ_0, each X_α is σ_0 (cf. [3]), consequently, each Y_α is σ_0. Since f is perfect,[2)] $f(\mathfrak{B})$ is a σ-locally finite net for Y (cf. [5]). Hence, if we put $\mathfrak{C}_n=\{Y_\alpha \mid \alpha \in \mathfrak{A}_n\}$ and $\mathfrak{C}=\bigcup_{n=1}^\infty \mathfrak{C}_n$, then $\mathfrak{C}$ is a σ-locally finite net for Y such that each $C \in \mathfrak{C}$ is σ_0. Therefore, Y is also a σ_0 space by Lemma 2.

3. **Proof of Theorem 1.** Let Y_1 be the aggregate of all points y in Y such that $\partial f^{-1}(y)$ is empty, Y_2 the aggregate of all points y in Y such that $\partial f^{-1}(y)$ is compact and non-empty and Y_3 the aggregate of all points y in Y such that $\partial f^{-1}(y)$ is not compact. Then we have $Y=Y_1 \cup Y_2 \cup Y_3$.

For each point $y \in Y_1$ select a point $x(y)$ of $f^{-1}(y)$ and let X_1 be the aggregate of all points $x(y)$ in X. Then $f(X_1)=Y_1$. Since $f \mid f^{-1}(Y_1)$ is closed and X_1 is closed in $f^{-1}(Y_1)$, f maps X_1 onto Y_1 homeomorphically.

Since $\partial f^{-1}(y)$ is non-empty, compact σ-subspace of X for each $y \in Y_2$, it is a compact, metrizable subspace (cf. [5]) and, consequently, it is not dense-in-itself by the assumption of f. And $X_2 =\bigcup\{\partial f^{-1}(y) \mid y \in Y_2\}$ is a σ-space (cf. [5]) and $f \mid X_2$ is a perfect mapping[2)] of X_2 onto Y_2. Hence, as in the proof of Lemma 3 we can see that Y_2 is the countable sum of subspaces, each of which is homeomorphic to a subspace of X.

Finally, Y_3 is σ-discrete in Y by Theorem. Therefore, Y_3 is

2) We shall say that f is *perfect* if it is a closed mapping such that $f^{-1}(y)$ is compact for each $y \in Y$.

a countable sum of subspaces, each of which is homeomorphic to a subspace of X. This completes the proof.

Proof of Theorem 2. Let $Y_1=Y-N, X_1=f^{-1}(Y_1)$ and $f_1=f|X_1$. Since X is hereditarily paracompact (cf. [5]), X_1 is also a collection-wise normal σ_0 subspace (cf. [3]). By Theorem 1 Y_1 is the countable sum of subspaces of Y, each of which is homeomorphic to a subspace of X. Since X_1 is a σ_0 space, Y_1 is a σ_0 space, too. Consequently, N must be uncountable-dimensional, completing the proof.

Proof of Theorem 3. Let us put Y_1, Y_2, Y_3, X_1, and X_2 as in Proof of Theorem 1; that is, Y_1 is the aggregate of all points y in Y such that $\partial f^{-1}(y)$ is empty, Y_2 the aggregate of all point y in Y such that $\partial f^{-1}(y)$ is compact and not dense-in-itself, and Y_3 is the aggregate of all points y in Y such that $\partial f^{-1}(y)$ is not compact, Then we have

$$Y_0=Y-Y_1\cup Y_2\cup Y_3.$$

Since f_1 maps X_1 onto Y_1 homeomorphically, we have $\dim Y_1=\dim X_1 \leqq 0$ (cf. [3]) and $f_2=f|X_2$ is a closed mapping of X_2 onto Y_2 such that $f_2^{-1}(y)=\partial f^{-1}(y)$ is compact and not dense-in-itself for each $y\in Y_2$. Since X_2 is σ_0, Y_2 is also σ_0 by Lemma 3. Finally, Y_3 is σ-discrete in Y by Theorem, therefore, σ_0. By the assumption that Y is not σ_0 Y_0 must be uncountable-dimensional. This completes the proof.

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