101. On the Nörlund Summability of Fourier Series and its Conjugate Series

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(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1968)

§ 1. Let $\{p_n\}$ be a sequence such that $P_n = p_0 + p_1 + \cdots + p_n \neq 0$ for $n = 0, 1, 2, \cdots$. A series $\sum_{n=0}^{\infty} a_n$ with its partial sum s_n is said to be summable (N, p_n) to sum s, if

$$\frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k \to s \quad \text{as} \quad n \to \infty.$$

Let $f(t)$ be a periodic finite-valued function with period 2π and integrable (L) over $(-\pi, \pi)$. Let its Fourier series be

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t). \tag{1.1}$$

Then the conjugate series of (1.1) is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t). \tag{1.2}$$

Throughout this paper, we write

$$\varphi(t) \equiv \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}, \qquad \Phi(t) \equiv \int_0^t |\varphi(u)|\, du,$$

$$\psi(t) \equiv \frac{1}{2}\{f(x+t) - f(x-t)\}, \qquad \Psi(t) \equiv \int_0^t |\psi(u)|\, du$$

and $\tau = [1/t]$, where $[\lambda]$ is the integral part of λ.

The purpose of this paper is to prove the following two theorems.

Theorem 1. *Let $\{p_n\}$ be a sequence such that*

$$p_n > 0,\ p_n \downarrow \quad \text{and} \quad P_n \to \infty. \tag{1.3}$$

And let $\lambda(t)$ be a positive integrable function such that

$$\int_\eta^n \frac{\lambda(u)}{u}\, du = O(P_n) \quad \text{as} \quad n \to \infty, \tag{1.4}$$

for any fixed $\eta > 0$. If

$$\Phi(t) = o\left(t\lambda\left(\frac{1}{t}\right) / P_\tau\right) \quad \text{as} \quad t \to +0, \tag{1.5}$$

then the series $\sum_{n=0}^{\infty} A_n(x)$ is summable (N, p_n) to sum $f(x)$.

Theorem 2. *Let $\{p_n\}$ and $\lambda(t)$ be defined as in Theorem 1. If*

$$\Psi(t) = o\left(t\lambda\left(\frac{1}{t}\right) / P_\tau\right) \quad \text{as} \quad t \to +0, \tag{1.6}$$

then the series $\sum_{n=1}^{\infty} B_n(x)$ *is summable* (N, p_n) *to sum*

$$\tilde{f}(x) = -\frac{1}{\pi}\int_0^{\pi} \psi(u) \cot \frac{u}{2}\, du$$

provided that this integral exists as a Cauchy integral at origin.

§ 2. Before we prove these theorems, we give some remarks on these results. Assuming that

$$\lambda(t) > 0, \quad \lambda(t) \uparrow \quad \text{and} \quad \lambda(n) \log n = O(P_n),$$

we have, for any fixed $\eta > 0$,

$$\int_{\eta}^{n} \frac{\lambda(u)}{u} du \leqq \lambda(n)(\log n - \log \eta) = O(\lambda(n) \log n) = O(P_n),$$

which shows that (1.4) holds. Thus we see that Theorem 1 is a generalization of a theorem due to Hirokawa and Kayashima [1; Theorem 3] and Theorem 2 is a generalization of a theorem due to Dikshit [2]. On the other hand, if we set $\lambda(t) = tp(t)$, where $p(t) = p_n$ for $n \leqq t < n+1$, $n = 0, 1, 2, \cdots$, then we have, for any fixed $\eta > 0$,

$$\int_{\eta}^{n} \frac{\lambda(u)}{u} du = \int_{\eta}^{n} p(u) du \leqq \int_0^{n+1} p(u) du = P_n$$

and

$$p_\tau / P_\tau = t\lambda\left(\frac{1}{t}\right) / P_\tau.$$

Thus we see that Theorems 1 and 2 are generalizations of theorems due to Singh [4; Theorems 1 and 2]. By the way, we know, from the argument in § 3, that our Theorems 1 and 2 are contained in known results due to Rajagopal [3] and due to Hirokawa and Kayashima [1; Theorem 2], respectively. But, the conditions of our Theorems are simpler than those of Rajagopal and of Hirokawa and Kayashima.

§ 3. We shall now prove Theorem 1. For the proof we need the following theorem due to Rajagopal [3].

Theorem A. *Let $p(t)$ be a positive monotone non-increasing function such that*

$$P(t) = \int_0^t p(u)\, du \to \infty \quad \text{as} \quad t \to \infty.$$

And let $p_n = p(n)$ for a non-negative integer n. If, for some fixed $\delta > 0$,

$$\int_{1/n}^{\delta} \Phi(t) \frac{d}{dt} \frac{P(1/t)}{t} dt = o(P_n), \tag{3.1}$$

then the series $\sum_{n=0}^{\infty} A_n(x)$ *is summable* (N, p_n) *to sum* $f(x)$.

To prove Theorem 1, we first define $p(t)$ by

$$p(t) = p_n \quad \text{for} \quad n \leqq t < n+1, \quad n = 0, 1, 2, \cdots.$$

And define $P(t) = \int_0^t p(u) du$. Then we have, by our assumption,

$$P(t)\to\infty \quad \text{as} \quad t\to\infty.$$

Furthermore we have, by (1.3), (1.4), and (1.5), for some fixed $\delta>0$,

$$\int_{1/n}^{\delta} \Phi(t)\frac{d}{dt}\frac{P(1/t)}{t}dt=o\left(\int_{1/n}^{\delta}\frac{t\lambda(1/t)}{P_\tau}\frac{p(1/t)/t+P(1/t)}{t^2}dt\right)$$
$$=o\left(\int_{1/n}^{\delta}\frac{1}{t}\lambda\left(\frac{1}{t}\right)dt\right)$$
$$=o\left(\int_{1/\delta}^{n}\frac{\lambda(u)}{u}du\right)=o(P_n),$$

which shows that (3.1) holds. Thus, by Theorem A, the proof of Theorem 1 is completed.

If we use, in the above proof, a theorem due to Hirokawa and Kayashima [1; Theorem 2] instead of Theorem A, we see that Theorem 2 is similarly proved.

References

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