97. On Screenable Topological Spaces

By John GREEVER
Research Institute for Mathematical Sciences, Kyoto University
and Harvey Mudd College, Claremont, California, U. S. A.

(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1968)

In recent years a number of papers, notably [2], [7], and [13], have been at least partially concerned with screenability in topological spaces and the interrelations between various generalized compactness properties and screenability. In this note it is shown that both screenability and strong screenability are intermediate to, and different from, certain generalized Lindelöf properties introduced in [6]. Also it is proved that in screenable spaces, countable metacompactness, countable paracompactness and countable compactness are equivalent to metacompactness, paracompactness and compactness, respectively. The latter result generalizes theorems of Heath [7] and the author [6].

Throughout this paper, *no* separation axiom (e.g., the T_1-axiom) is assumed tacitly for the topological spaces under discussion. All terminology is consistent with that used in [4] and [6]. The properties of screenability and strong screenability were first defined by Bing [2].

Definition 1. A collection $\mathcal{C}$ of subsets of a topological space X is

(i) *σ-pairwise-disjoint* if and only if $\mathcal{C}$ is the union of countably many collections each of which is a pairwise-disjoint collection of subsets of X.

(ii) *discrete* if and only if $\{\bar{C}: C \in \mathcal{C}\}$ is pairwise-disjoint and the union of any subcollection of $\{\bar{C}: C \in \mathcal{C}\}$ is closed in X.

(iii) *σ-discrete* if and only if $\mathcal{C}$ is the union of countably many discrete collections of subsets of X.

Definition 2. A topological space X is

(i) *screenable* if and only if each open cover of X has a σ-pairwise-disjoint, open refinement.

(ii) *strongly screenable* if and only if each open cover of X has a σ-discrete, open refinement.

Lemma. *If $\mathcal{R}$ is a star-countable open cover of a topological*

This research was supported in part by Harvey Mudd College through a grant of sabbatical leave and fellowship funds.

space X, then $\mathcal{R}$ is σ-discrete.

Proof. Suppose $\{\mathcal{R}_\alpha : \alpha \in \Lambda\}$ is the collection of components of $\mathcal{R}$ and, for each $\alpha \in \Lambda$, let $X_\alpha = \cup \mathcal{R}_\alpha$. As observed in a proof of [6, Th. 1], each component of $\mathcal{R}$ is countable and $\{X_\alpha : \alpha \in \Lambda\}$ is a pairwise-disjoint collection of open and closed sets whose union is X. For each $\alpha \in \Lambda$, let $\mathcal{R}_\alpha = \{R_1^\alpha, R_2^\alpha, \cdots\}$, where $R_j^\alpha = \phi$ for $j > n$, if $\mathcal{R}_\alpha$ is a finite collection of n elements instead of a countably infinite collection. For each positive integer i, let $\mathcal{U}_i = \{R_i^\alpha : \alpha \in \Lambda\}$. Then $\mathcal{R} = \cup \{\mathcal{U}_i : i = 1, 2, \cdots\}$. Also, if i is a positive integer, then $R_i^\alpha \subset X_\alpha$ for each $\alpha \in \Lambda$. Since $\{X_\alpha : \alpha \in \Lambda\}$ is a pairwise-disjoint collection of open sets, it follows that $\mathcal{U}_i$ is a discrete collection. Thus $\mathcal{R}$ is σ-discrete.

Theorem 1. *Let X be a topological space. Then*

(i) *X is strongly screenable, if X is hypoLindelöf.*

(ii) *X is screenable, if X is strongly screenable.*

(iii) *X is metaLindelöf, if X is screenable.*

Proof. For (i) note that each open cover of a hypoLindelöf space has a star-countable open refinement which must be σ-discrete by the lemma. (ii) is immediate from the definitions involved as is (iii), since a σ-pairwise-disjoint collection of sets must be point-countable.

Examples 1 through 3 below show that the properties related by Theorem 1 are distinct in regular T_1-spaces. It is known that a regular space is hypoLindelöf if and only if it is hypocompact [14] and strongly screenable if and only if it is paracompact [11]. An affirmative answer to Problem 1, posed previously by Nagami [13], would also be an affirmative answer to a problem posed by Dowker [3] and Katětov [10], independently, as to whether there exists a normal T_1-space which is not countably paracompact.

Example 1. A strongly screenable, normal T_1-space which is not hypoLindelöf (but is countably hypocompact).

Construction. Since Bing [2] has shown that every metrizable space is strongly screenable, it suffices to recall that Example H of [6] is metrizable, and thus countably hypocompact, but not hypoLindelöf.

Example 2. A screenable, regular T_1-space which is not strongly screenable.

Construction. Bing's Example B in [2] has the requisite properties, as does the more recent, simpler Example 2 of Heath [7].

Problem 1. Does there exist a screenable normal T_1-space which is not strongly screenable?

Example 3. A metaLindelöf, normal T_1-space which is not screenable (but is metacompact and countably hypocompact).

Construction. In light of Theorem 2 below, Example I_1 of [6], due to Michael [12] as a modification of an example of Bing [2], has the requisite properties.

Theorem 2. *Suppose X is a screenable topological space. Then*

(i) *X is metacompact if X is countably metacompact.*

(ii) *X is paracompact if X is countably paracompact.*

(iii) *X is compact if X is countably compact.*

Proof. Let $\mathcal{U}$ be an open cover of X. Since X is screenable, there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \cdots$ each term of which is a pairwise-disjoint collection of open sets, such that $\cup_{i=1}^{\infty}\mathcal{U}_i$ is an open refinement of $\mathcal{U}$. For each positive integer n, let $H_n = \cup(\cup_{i=1}^{n}\mathcal{U}_i)$ and let $F_n = X - H_n$. Then $F_1, F_2, \cdots$ is a decreasing sequence of closed sets such that $\cap_{i=1}^{\infty} F_i = \phi$.

The existence of the sequences $G_1, G_2, \cdots$ described next is assured by theorems of F. Ishikawa [9] which characterize countable metacompactness and countable paracompactness in the manner indicated. If X is countably metacompact, let $G_1, G_2, \cdots$ be a decreasing sequence of open sets in X such that $\cap_{i=1}^{\infty} G_i = \phi$ and $F_n \subset G_n$ for each positive integer n.

Let $\mathcal{R} = \mathcal{U}_1 \cup \{R : R = G_{i-1} \cap U,\ U \in \mathcal{U}_i,\ i \geq 2\}$. Then $\mathcal{R}$ is a collection of open sets each one of which is contained in some element of $\mathcal{U}$. Suppose $x \in X$. There exists a least positive integer M such that $x \in H_M$. If $M=1$, then $x \in H_1 = \cup\mathcal{U}_1$. If $M>1$, then $x \in H_M - H_{M-1}$ and so $x \in F_{M-1} \subset G_{M-1}$ and $x \in \cup\mathcal{U}_M$; thus $x \in G_{M-1} \cap U$ for some $U \in \mathcal{U}_M$. In either case, $x \in \cup\mathcal{R}$ and so $\mathcal{R}$ is an open cover of X.

Now suppose $p \in X$. If X is countably metacompact, let N be a positive integer such that $p \notin G_j$ whenever $j \geq N$. Thus $p \notin \cup\{R : R = G_{i-1} \cap U,\ U \in \mathcal{U}_i,\ i > N\}$ and p belongs to at most one element of each of $\mathcal{U}_1, \cdots, \mathcal{U}_N$. Thus p belongs to at most N elements of $\mathcal{R}$ and hence $\mathcal{R}$ is point-finite. This proves that X is metacompact. If X is countably paracompact, let N be a positive integer such that $p \notin \bar{G}_j$ whenever $j \geq N$ and let $V_0 = X - \bar{G}_N$. Then $V_0 \cap G_j = \phi$ if $j \geq N$ and so $V_0 \cap \{R : R = G_{i-1} \cap U,\ U \in \mathcal{U}_i,\ i > N\} = \phi$. Let V be the intersection of V_0 with the (at most N) elements of $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_N$ to which p belongs. Then V is an open set about p which intersects at most N elements of $\mathcal{R}$ and so $\mathcal{R}$ is locally-finite. This proves that X is paracompact.

Finally, if X is countably compact then X is countably metacompact and so is metacompact by the above proof. It follows from a theorem of Arens and Dugundgi [1], which as noted in [6, pp. 39–49] does not require a T_1-hypothesis, that X is compact.

Corollary 1. *Suppose X is a screenable topological space in which every closed set is a G_δ. Then X is metacompact.*

Proof. In [5] it was shown that a space in which every closed set is a G_δ must be countably metacompact.

Problem 2. Does there exist a screenable, regular T_1-space which is not metacompact?

Corollary 2. *Suppose X is a screenable, normal topological space in which every closed set is a G_δ. Then X is paracompact and countably hypocompact.*

Proof. As in Corollary 1, X must be countably metacompact. Dowker [3] and Katětov [10], independently, have shown that normal, countably metacompact spaces are countably paracompact and Iséki [8] showed that normal, countably paracompact spaces are countably hypocompact. Finally, X is paracompact by Theorem 2.

Corollary 1 was proved first by Heath [7] whose method of proof suggested Theorem 2. In [6] a proof was given of Theorem 2 with "screenable" replaced by "hypoLindelöf" and it was also shown that every hypoLindelöf, countably hypocompact space is hypocompact. Example 1 above shows that not every screenable (or strongly screenable), countably hypocompact space is hypocompact. Theorem 2 is also not valid if "screenable" is replaced with "metaLindelöf," as is demonstrated by Example 3 above.

References

[1] Richard Arens and J. Dugundji: Remark on the concept of compactness. Portugaliae Math., **9**, 141–143 (1950).

[2] R. H. Bing: Metrization of topological spaces. Canad. J. Math., **3**, 175–186 (1951).

[3] C. H. Dowker: On countably paracompact spaces. Canad. J. Math., **3**, 219–224 (1951).

[4] John Greever: Theory and Examples of Point-set Topology. Brooks-Cole, Belmont, Calif. (1967).

[5] ——: On spaces in which every closed set is a G_δ Proc. Japan Acad., **43**, 445–447 (1967).

[6] ——: On some generalized compactness properties. Publ. Res. Inst. Math. Sci. Kyoto Univ. Ser. A, **4**, 39–49 (1968).

[7] R. W. Heath: Screenability, pointwise paracompactness, and metrization of Moore spaces. Canad. J. Math., **16**, 763–770 (1964).

[8] Kiyoshi Iséki: A note on countably paracompact spaces. Proc. Japan Acad., **30**, 350–351 (1955).

[9] Fumie Ishikawa: On countably paracompact spaces. Proc. Japan Acad., **31**, 686–687 (1955).

[10] Miroslav Katětov: On real-valued functions in topological spaces. Fund. Math., **38**, 85–91 (1951).

[11] Ernest Michael: A note on paracompact spaces. Proc. Amer. Math. Soc., **4**, 831–838 (1953).

[12] ——: Point-finite and locally finite coverings. Canad. J. Math., **7**, 275–279 (1955).

[13] Keiô Nagami: Paracompactness and strong screenability. Nagoya Math. J., **8**, 83–88 (1955).
[14] Ю. М. Смирнов: О сильно-паракомпактных ироcтранствах. Изв. Акад. Наук СССР., сер. мат., **20**, 253–274 (1956).