# 148. The Completion of a Convergence Space in the Sense of H. R. Fisher

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In this paper we shall make a study of the completion of a space: here by a space we mean a set in which there is defined a closure operation satisfying three conditions $A\subseteq\overline{A}$, $\overline{\phi}=\phi$, and $\overline{A\cup B}=\overline{A}\cup\overline{B}$.

Such a space was introduced by Tukey [8] and studied also by Fisher [4] under the name of a convergence space.[1]

In this paper we shall describe a space by assigning a neighborhood system to each point of it.

Thus we get a generalization of the results of the author's paper [7].[2]

§1. Let $\varphi$ be a mapping of a set $X$ into a set $Y$. Then for a family $\mathfrak{A}$ consisting of subsets of $X$, we will denote by $\varphi(\mathfrak{A})$ the family $\{\varphi(A)\mid A\in\mathfrak{A}\}$ and for a family $\mathfrak{B}$ consisting of subsets of $Y$, let's denote by $\varphi^{-1}(\mathfrak{B})$ the family $\{\varphi^{-1}(B)\mid B\in\mathfrak{B}\}$.

Let $X$ be a subset of a set $X^*$, then for a filter $\mathfrak{f}$ in $X$, the filter in $X^*$ generated by $\mathfrak{f}$ is denoted by $\mathfrak{f}^*$.

We consider a set $X$ together with a family $N$ of filters in $X$ satisfying the following three conditions:

N1) to every $x\in X$ there corresponds uniquely a filter $\mathfrak{N}(x)$ each member of which contains $x$,

N2) a filter in $X$ containing an element of $N$ also belongs to $N$,

N3) for every $x\in X$, $\mathfrak{N}(x)\in N$.

We will denote such a space $X$ with $N$ by $(X;N)$ and call it a *space* simply.

A filter base $\mathfrak{f}$ in $X$ *converges* to $x$ in $X$ if and only if the filter generated by $\mathfrak{f}$ contains $\mathfrak{N}(x)$.[3]

A filter $\mathfrak{N}(x)$ and each of its members are called the *neighborhood system* of $x$ and a *neighborhood* of $x$ respectively.

A mapping $\varphi$ of a space $(X;N)$ into a space $(Y;M)$ is *continuous* if and only if for every $x\in X$ a filter generated by $\varphi(\mathfrak{N}(x))$ contains

1) In this paper spaces are all $\mathcal{T}_1$ convergence spaces. See [4].

2) In that paper [7] the condition C6) is stated erroneously. It must be read as C6) of this paper and $\mathfrak{f}$ in the last two lines on page 464 must be a leg.

3) N1) with this definition of convergence is called $\mathcal{T}_1$ *convergence structure* of a space by Fisher.

$\mathfrak{N}(\varphi(x))$; furthermore, if $\varphi$ is one-to-one and onto and if $\varphi^{-1}$ is also continuous, $\varphi$ is called a *homeomorphism*.

The completeness is defined as follows: $(X; N)$ is *complete* if and only if every filter belonging to $N$ converges to some point of $X$ in $X$.

If $(X; N)$ is complete then $N$ is uniquely determined by the neighborhood systems of all the points of $X$.

A filter $\mathfrak{f} \in N$ is *minimal* if there is no filter belonging to $N$ that is contained properly in $\mathfrak{f}$.

A filter $\mathfrak{f} \in N$ is a *leg* in $X$ if it converges to no point of $X$ in $X$.

A *completion* $(X^*; N^*)$ of a space $(X; N)$ is such a space that satisfies, in addition to the conditions N1), N2, and N3), the conditions C1) to C6) below:

C1) $X^* \supseteq X$,

C2) $(X^*; N^*)$ is complete,

C3) to every subset $V$ of $X$, there corresponds a subset $V^*$ of $X^*$ such that $V^* \cap X = V$, and for every $x \in X$, $\{S \mid S \supseteq V^*, V \in \mathfrak{N}(x)\}$ is the neighborhood system of $x$ in $X^*$,

C4) for a point $x \in X^* \sim X$ and a subset $V$ of $X$, if every leg $\mathfrak{f} \in N$ converging to $x$ in $X^*$ contains $V$ then $x \in V^*$, and $\{S \mid S \supseteq V^* \ni x, V \subseteq X\}$ is the neighborhood system of $x \in X^* \sim X$ in $X^*$,

C5) if $\mathfrak{g} \in N^*$ then $\{V \mid V^* \in \mathfrak{g}, V \subseteq X\} \in N$,

C6) any leg in $X$ converges to only one point in $X^*$, and for every $x \in X^* \sim X$, there exists at least one leg in $X$ converging to $x$ in $X^*$.

Let's denote by $\mathfrak{N}^*(x)$ the neighborhood system of a point $x \in X^*$ in $X^*$.

Now, assume that there exists a completion $(X^*; N^*)$ of a space $(X; N)$.

Then for any two legs $\mathfrak{f}$ and $\mathfrak{g}$, if $\mathfrak{f} \supseteq \mathfrak{g}$ and $\mathfrak{f}$ converges to $x$ in $X^*$, then by C6), $\mathfrak{g}$ also converges to $x$ in $X^*$. So the filter $\bigcap_{\mathfrak{g} \subseteq \mathfrak{f}, \mathfrak{g} \in N} \mathfrak{g}$ converges to $x$ in $X^*$. Clearly it must belong to $N$ by N3) and C5), and so it is a leg.

Denote $\bigcap_{\mathfrak{g} \subseteq \mathfrak{f}, \mathfrak{g} \in N} \mathfrak{g}$ by $[\mathfrak{f}]$. Thus the following holds.

E) If $\mathfrak{f}$ is a leg then $[\mathfrak{f}]$ is also a leg.

Next, suppose that a space $(X; N)$ satisfies the above Condition E).

A leg $\mathfrak{f}$ is minimal if and only if $\mathfrak{f} = [\mathfrak{f}]$.

Put, for every subset $V$ of $X$,

$$V^* = V \cup \{\mathfrak{f} \mid \mathfrak{f} \text{ is a minimal leg such that } V \in \mathfrak{f}\}.$$

For every minimal leg $\mathfrak{f}$, let $\mathfrak{N}^*(\mathfrak{f})$ be the filter in $X^*$ generated by the filter base $\{V^* \mid V \in \mathfrak{f}\}$. On the other hand, for all $x \in X$, put

$$\mathfrak{N}^*(x) = \{S \mid S \supseteq V^*, V \in \mathfrak{N}(x)\} \text{ and}$$

$N^* = \{\mathfrak{f} \mid \mathfrak{f}$ is a filter in $X^*$ such that $\{V \mid V^* \in \mathfrak{f}, V \subseteq X\} \in N\}$.

Thus we get a space $(X^*; N^*)$.

Then, an arbitrary $\mathfrak{f} \in N^*$ converges to some point $x \in X^*$ in $X^*$. Because, if $\{V \mid V^* \in \mathfrak{f}\}$ converges to some point $y \in X$ in $X$ then we can take $x=y$; on the other hand, if $\{V \mid V^* \in \mathfrak{f}\}$ is a leg then we have $x=[\{V \mid V^* \in \mathfrak{f}\}]$.

Thus $(X^*; N^*)$ is complete.

Suppose that a leg $\mathfrak{f} \in N$ in $X$ converges to $\mathfrak{g} \in N$ in $X^*$. Then $\mathfrak{f}^*$ contains the filter in $X^*$ generated by $\{V^* \mid V \in \mathfrak{g}\}$. So $\mathfrak{f} \supseteq \mathfrak{g}$, hence $[\mathfrak{f}]=\mathfrak{g}$. And no leg converges to any point of $X$ in $X^*$.

This result shows that $(X^*; N^*)$ satisfies the former part of C6).

The other conditions for the completion are satisfied almost clearly. Thus we get,

**Theorem.** *There exists a completion $(X^*; N^*)$ of a space $(X; N)$ if and only if for every leg $\mathfrak{f}$ in $X$, $[\mathfrak{f}]$ is also a leg.*

Let $(X^*; N^*)$ and $(X^+; N^+)$ be two completions of a space $(X; N)$.

For every $x \in X^* \sim X$, there exists a leg $\mathfrak{f} \in N$ that converges to $x$ in $X^*$, and $\mathfrak{f}$ converges to some point $y$ in $X^+$. Let us put $\varphi(x)=y$. Furthermore we put $\varphi(x)=x$ for all $x \in X$. Then we get a one-to-one mapping $\varphi$ of $X^*$ onto $X^+$.

From Condition C4) it follows that for every subset $V$ of $X$, $\varphi(V^*)=V^+$.

Thus we get

**Theorem.** *A completion of a space $(X; N)$ is uniquely determined by $(X; N)$.*

**§ 2.** A *product* $(X; N)$ of spaces $(X_\lambda; N_\lambda)$, $\lambda \in \Delta$ is defined to be a space satisfying Conditions P1) to P3) below:

P1) $X=\Pi_{\lambda \in \Delta} X_\lambda$, i.e., $X$ is the Cartesian product of $X_\lambda$,

P2) a filter $\mathfrak{f}$ in $X$ belongs to $N$ if and only if $P_\lambda(\mathfrak{f}) \in N_\lambda$ for all $\lambda \in \Delta$, where $P_\lambda$ is the projection of $X$ onto its $\lambda$-component $X_\lambda$,

P3) for all $x_\lambda \in X_\lambda$, the neighborhood system $\mathfrak{N}(x)$ of $x=\Pi_{\lambda \in \Delta} x_\lambda \in X$ in $X$ is the least filter containing $\cup_{\lambda \in \Delta} P_\lambda^{-1}(\mathfrak{N}(x_\lambda))$.

P3) is equivalent to:

P3)′ the neighborhood system of $\Pi_{\lambda \in \Delta} x_\lambda \in \Pi_{\lambda \in \Delta} X_\lambda$ in $\Pi_{\lambda \in \Delta} X_\lambda$ is the least filter of which the projection into the $\lambda$-component $X_\lambda$ agrees with the neighborhood system $\mathfrak{N}(x_\lambda)$ of $x_\lambda$ in $X_\lambda$ for all $\lambda \in \Delta$.[4]

A filter $\mathfrak{f}$ in $X$ converges to a point $x=\Pi_{\lambda \in \Delta} x_\lambda$ if and only if the filter $P_\lambda(\mathfrak{f})$ in $X_\lambda$ converges to $x_\lambda$ in $X_\lambda$ for all $\lambda \in \Delta$.

Hence we have

**Theorem.** *A product $\Pi(X_\lambda; N_\lambda)$ is complete if and only if each factor space $(X_\lambda; N_\lambda)$ is complete.*

4) Furthermore it is equivalent to: P3)″ a product of spaces $(X_\lambda; N_\lambda)$ has the weakest $\mathfrak{T}_1$ convergence structure such that each projection onto a component is continuous.

The following Condition $T_2$) is known as the Hausdorff separation axiom:

$T_2$) for any distinct points $x$ and $y$ there exist disjoint neighborhoods of $x$ and $y$.

If a space $(X;N)$ satisfies the above Condition $T_2$) then $(X;N)$ is said to be a *$T_2$ space*.[5)]

A product $\Pi(X_\lambda;N_\lambda)$ is a $T_2$ space if and only if each factor space $(X_\lambda;N_\lambda)$ is a $T_2$ space.

Suppose that $T_2$ spaces $(X_\lambda;N_\lambda)$, $\lambda\in\Delta$ have completions $(X_\lambda^*;N_\lambda^*)$ which are also $T_2$.

Then, the product $\Pi(X_\lambda;N_\lambda)$ of $(X_\lambda;N_\lambda)$ satisfies our Condition E) and so has the completion.

Denote $\Pi_{\lambda\in\Delta}(X_\lambda;N_\lambda)$ by $(X;N)$.

A filter $\mathfrak{f}\in N$ is a leg if and only if $P_\lambda(\mathfrak{f})$ is a leg for at least one $\lambda\in\Delta$.

Let us define a mapping $\varphi$ of $(X^*;N^*)$ into $\Pi(X_\lambda^*;N_\lambda^*)$ such that $\varphi|X$ is the identity and for every $x\in X^*\sim X$, $\varphi(x)$ is a point to which a leg $\mathfrak{f}\in N$ converging to $x$ in $X^*$ converges in $\Pi(X_\lambda^*;N_\lambda^*)$. Then $\varphi$ is clearly one-to-one and onto. We have $\varphi((P_\lambda^{-1}(K))^*)\subseteq P_\lambda^{*-1}(K^*)$ for any subset $K$ of $X_\lambda$, where $P_\lambda^*$ is the projection of $\Pi(X_\lambda^*;N_\lambda^*)$ onto the $\lambda$-component $(X_\lambda^*;N_\lambda^*)$.

Thus $\varphi$ is continuous.

Let $\mathfrak{f}_\lambda$ be any minimal filter belonging to $N_\lambda$ and $K$ be any subset of $X_\lambda$ belonging to $\mathfrak{f}_\lambda$.

If $\varphi^{-1}$ is continuous then $\varphi((P_\lambda^{-1}(K))^*)\supseteq P_\lambda^*(M^*)$ for some $M\in\mathfrak{f}_\lambda$.

In view of the proposition below, we see that $(X_\lambda^*;N_\lambda^*)$ for $\lambda\neq\mu$ are all Hausdorff spaces if a non-complete factor space $(X_\mu;N_\mu)$, $\mu\in\Delta$ exists.

**Proposition.** *Let $(X^*;N^*)$ be the completion of a topological space $(X;N)$. Then $(X^*;N^*)$ is also a topological space if and only if for every minimal leg $\mathfrak{f}\in N$ and for any element $V$ of $\mathfrak{f}$, there exists some element $W\in\mathfrak{f}$ such that $V\in\mathfrak{N}(x)$ for any $x\in W$.*[6)]

Conversely, for these $(X_\lambda;N_\lambda)$, $\lambda\in\Delta$, if $(X_\lambda^*;N_\lambda^*)$ is a Hausdorff space whenever there is a non-complete space $(X_\mu;N_\mu)$, $\mu\in\Delta$ and $\lambda\neq\mu$, then for every finite subset $\Gamma\subseteq\Delta$ and for every open set $K_\lambda\subseteq X_\lambda$, $\lambda\in\Delta$,

$$\varphi((\Pi_{\lambda\in\Gamma}K_\lambda\cdot\Pi_{\lambda\notin\Gamma}X_\lambda)^*)=\Pi_{\lambda\in\Gamma}K_\lambda^*\cdot\Pi_{\lambda\notin\Gamma}X_\lambda^*.\text{[7)]}$$

5) It is to be noted that the condition $\overline{\overline{A}}=\overline{A}$ is not assumed in this paper.

6) This proposition is obtained directly from the following well known remark. **Remark.** A space $(X;N)$ is a topological space if and only if for every $x\in X$ and its any neighborhood $V\in\mathfrak{N}(x)$ there exists a neighborhood $W\in\mathfrak{N}(x)$ such that for all points $y\in W$, $V\in\mathfrak{N}(y)$.

7) When a non-complete factor space is only one $(X_\mu;N_\mu)$, $K_\mu$ may be arbitrary subset of $X_\mu$.

Thus, the following holds.

**Theorem.** *Let* $(X_\lambda; N_\lambda)$, $\lambda \in \Delta$ *be* $T_2$ *spaces such that they have the completions* $(X_\lambda^*; N_\lambda^*)$ *which are also* $T_2$. *Then there exists a completion of the product* $\Pi(X_\lambda; N_\lambda)$. *And it agrees with* $\Pi(X_\lambda^*; N_\lambda^*)$ *if and only if* $(X_\lambda^*; N_\lambda^*)$ *is a Hausdorff space whenever there exists a non-complete space* $(X_\mu; N_\mu)$, $\mu \in \Delta$ *and* $\mu \neq \lambda$.

Let $(X^*; N^*)$ be the completion of $(X; N)$.

In general, for every subsets $S$ and $U$, if $S \cap U = \phi$ then $S^* \cap U^* = \phi$.

If there are two legs $\mathfrak{f}$ and $\mathfrak{g}$ such as for any $V \in \mathfrak{f}$ and for any $W \in \mathfrak{g}$, $V \cap W \neq \phi$, then $\{S \mid S \supseteq V \cap W, V \in \mathfrak{f}, W \in \mathfrak{g}\}$ is also in $N$. Either this filter converges in $X$ or it is a leg, If it is a leg, then $[\mathfrak{f}] = [\{S \mid S \supseteq V \cap W, V \in \mathfrak{f}, W \in \mathfrak{g}\}] = [\mathfrak{g}]$.

So we get

**Proposition.** *A completion* $(X^*; N^*)$ *of a* $T_2$ *space* $(X; N)$ *is* $T_2$ *if and only if for every leg* $\mathfrak{f}$ *and for all points* $x \in X$, $V \cap W = \phi$ *for some subset* $V \in [\mathfrak{f}]$ *of* $X$ *and some neighborhood* $W$ *of* $x$.

**§ 3.** A continuous mapping $\varphi$ of $(X; N)$ into $(Y; M)$ is *$*$-continuous* if a filter generated by $\varphi(\mathfrak{f})$ belongs to $M$ for every filter $\mathfrak{f} \in N$.

Let $\varphi$ be a $*$-continuous mapping of a space $(X; N)$ into a space $(Y; M)$.

If $(X; N)$ and $(Y; M)$ have the completions $(X^*; N^*)$ and $(Y^*; M^*)$, then there exists a mapping $F$ of $(X^*; N^*)$ into $(Y^*; M^*)$ such that for every $x \in X^* \sim X$, $F(x)$ is a point of $Y^*$ to which an image $\varphi(\mathfrak{f})$ of a minimal leg $\mathfrak{f}$ converging to $x$ in $X^*$ converges and $F \mid X = \varphi$. We will call this mapping $F$ an *extention* of $\varphi$.

**Proposition.** *Let a mapping* $F$ *of a completion* $(X^*; N^*)$ *of a spaces* $(X; N)$ *into a complete spaces* $(Y; M)$ *be an extention of a* $*$*-continuous mapping* $\varphi$ *of* $(X; N)$ *into* $(Y; M)$. *Then* $F$ *is continuous if for any* $y \in F(X^*)$, $\{S \mid S \supseteq \overline{V} \cap F(X^* \sim X), V \in \mathfrak{N}(y)\} \supseteq \mathfrak{N}(y)$.

## References


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