144. A Remark on a Problem of M. A. Naimark

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Gelfand and Naimark [6] characterized the algebra of all continuous complex-valued functions on a compact Hausdorff space as a commutative Banach *-algebra which satisfies the condition $||xx^*|| = ||x^*|| \cdot ||x||$: while Aren's generalization of the Gelfand-Naimark theorem is that a complete commutative seminormed *-algebra with a partition of unity is equivalent to the algebra of all continuous complex-valued functions on a locally compact paracompact space $C(T, K)$ [1]. A question is posed by Naimark in his treatise [6]: Is it possible to characterize all complete commutative seminormed *-algebras which are equivalent to (topologically equivalent to algebraically *-isomorphic) the algebras of all continuous complex-valued functions on locally compact Hausdorff spaces? Even though some more general result in this direction was obtained by Sha [6, 1964], it seems the problem remains open. Incidentally a solution of the problem was given by the writer [8, p. 182]. The purpose of this note is to present a modified proof of the solution and a second characterization in terms of seminorms.

"Seminormed algebra" and "locally multiplicatively convex algebra" (LMC) will be used synonymically in this paper. A subset Σ of an algebra is said to multiplicatively convex (m-convex) if $\Sigma\Sigma \subset \Sigma$. We assume the family $\mathcal{V}$ of seminorms of an algebra is so large that $V \in \mathcal{V}$, $U \leq V$ imply $U \in \mathcal{V}$. Some basic theorems and definitions employed hereafter are referred to [1], [3], and [4].

1. Functional representation. Lemma. If βX is the Stone-Čech compactification of a completely regular space X, then any unbounded continuous real function of f on X can be continuously extended to an extended function $\bar{f}$ over βX which admits $+\infty$ or $-\infty$ on some subsets of $\beta X - X$.

First proof. Let B be the two-point $(\pm\infty)$ compactification of the real line. Then B is a compact Hausdorff space and f is a continuous mapping from X into B. By the Stone-Čech compactification theorem [2, p. 153] f has a continuous extension $\bar{f}$ on βX and the lemma is proved.

Second proof. Suppose

$$f_n(x)=\begin{cases} n & f(x)\geq n, \\ f(x) \quad \text{if} & -n<f(x)<n, \\ -n & f(x)\leq -n. \end{cases}$$

Let $p\in \beta X-X$ and

$$N_n(p)=\{x : |\bar{f}_n(x)-\bar{f}_n(p)|<\varepsilon<|,\ x\in X\}$$

where $\bar{f}_n$ is the continuous extension of f_n over βX. In case f is bounded on $N_n(p)\cap X$, f and f_m coincide on $N_n(p)\cap X$ for some m and f can be continuously extended to a finite value at p. Otherwise f is unbounded on $N_n(p)\cap X$ for all n and ε and $f(x)$ is either $>n$ or $<-n$ on $N_n(p)\cap X$ for all large n. Then $\bar{f}$ assumes $+\infty$ or $-\infty$ at p.

Theorem 1. A complete commutative seminored *-algebra A with unity and satisfying the condition:

$$V(xx^*)\geq k_V V(x)V(x^*), \quad V\in \mathcal{V}, \quad x\in A,$$

is equivalent to the algebra $C(T_0, K)$, with τ_0-topology$\leq k$-topology (compact-open), of all continuous complex functions on a completely regular space T_0.

Proof. The algebra A is equivalent to a subalgebra S of $C(T, K)$, where $T=\bigcup_{V\in\mathcal{V}} Q_V$ is the union of mutually disjoint compact sets Q_V [8, p. 178]. We denote $V(x)\geq U(x)$ for all $x\in A$ by $V\geq U$. A function $f\in C(T, K)$ belongs to S if and only if $f_V(M_V)=f_U(M_U)$ for all $V\geq U$, f_V being the restriction of f on Q_V, $M_U=\pi M_V$ for $M_V\in Q_V$ and the natural projection from Q_V to Q_U.

It follows from the lemma that each $f\in S$ has a continuous extension $\bar{f}$ over βT. Denote by $\bar{S}$ the set of all $\bar{f}$ for $f\in S$ and let L be the class of subsets L_m of βT defined by $L_m=\{t;\ \bar{f}\in\bar{S}$ implies $\bar{f}(t)=\bar{f}(m),\ t\in\beta T\}$, $(m\in\beta T)$. A subset F of L is said to be closed if and only if the union of $L_m\in F$ is closed in βT. The mapping $\sigma:\beta T\rightarrow L$ is continuous and the σ-images $\bar{Q}'_V$ of $\bar{Q}_V$ are compact sets in the topological space T_0, the σ-image of T. The subalgebra S of $C(T, K)$ is a subalgebra S' of $C(T_0, K)$; while S' endowed with the uniform topology on the compact sets $\bar{Q}'_V$ is equivalent to the algebra A.

On the other hand an arbitrary continuous function f' on T_0 is continuous on T and satisfies the condition: $f_V(M_V)=f_U(M_U)$ if $V\geq U$ and $M_U=\pi M_V$. Then $f'\in S'$ and $S'=C(T_0, K)$.

Definition 1. Let $\mathcal{M}$ be the set of all closed maximal ideals M in a topological algebra A. The topology of $C(\mathcal{M}, K)$ defined by the uniform convergence on the closed equicontinuous subsets of $\mathcal{M}$ is called *Michael's topology.*

Corollary 1 (Michael). A complete commutative *-algebra A with unity e is equivalent to $C(\mathcal{M}, K)$ under Michael's topology.

Proof. There is one to one correspondence between the closed maximal ideals of A and the continuous multiplicative linear functionals f on A satisfying the condition $f(e)=1$, and also to each seminorm V there associates a closed, convex, symmetric, and radial at 0 subsets $C_V=\{x: |V(x)|\leq 1, x \in A\}$ of A [4]. A closed set F of continuous multiplicative linear functionals f corresponding to the closed maximal ideals in Q_V is equicontinuous on account of $|f(x)|\leq f(e)V(x)\leq 1$ for $x \in C_V$, $f \in F$. Conversely, a closed set F of continuous multiplicative linear functionals defined by $\{f: |f(x)|\leq 1, f(e)=1, x \in C_V\}$ for some closed, convex, symmetric, and radial at 0 set C_V in A associated with a seminorm V is a closed subset G of $\bar{Q}_V$. The topology of uniform convergence on the closed equicontinuous set F is the same as the topology defined by the seminorm on G.

Definition 2. A *m-barrel* in a LMC algebra is a barrel which is m-convex. A LMC algebra is called *m-barrelled* if every m-barrel is a neighborhood of 0.

The following is a consequence of an observation that to each seminorm there corresponds a compact set in T and conversely (see Proof of Theorem 1).

Corollary 2. A complete commutative seminormed *-algebra A is equivalent to $C(T_0, K)$ of all continuous complex function on a completely regular space T_0 under k-topology if and only if A is m-barrelled.

2. Naimark's problem. Theorem 2. The necessary and sufficient condition that a complete seminormed commutative *-algebra A satisfying: $V(xx^*)\geq k_V V(x)V(x^*)$, $V \in \mathcal{V}$, be equivalent to $C(T, k)$, with k-topology, of all continuous complex functions on a locally compact Hausdroff space T_0 is:

To any closed maximal ideal M_0 in A, there exist $x_0 \in M_0$ and $\varepsilon>0$ such that all closed maximal ideals M satisfying $|x_0(M)|\leq\varepsilon$ contain a kernel ideal E.

Proof. *Necessity.* Let T be a locally compact Hausdorff space and A be equivalent to $C(T, K)$ under k-topology. $M_0 \in T$ has an open neighborhood N with compact closure $\bar{N}$. There exists a real function $x_0 \in C(T, K)$ with $x_0(M_0)=0$ and $x_0(M)=1$ for $M \in CN$. The set $G=\{M; |x(M)|\leq\varepsilon<1\}$ is compact and is the support of a seminorm V. Since there is one to one correspondence between the closed maximal ideals in $C(T, K)$ and the points in T_0 [5, p. 325], the continuous functions vanishing on G constitute the kernelideal of the seminorm V_0, contained in all closed maximal ideals M which satisfy the condition $|x_0(M)|\leq\varepsilon$.

Sufficiency. It suffices to prove the local compactness of the space T_0. As E is the kernel ideal of a seminorm V_0, contained in all closed maximal ideals M satisying $|x(M)|\leq\varepsilon$, the set $\gamma=\{M:|x(M)|\leq\varepsilon, M\in\bar{Q}_{V_0}\}$ consists of all closed maximal ideals in A subjected to the condition $|x(M)|\leq\varepsilon$. Let σ be the same mapping from the space T to T_0 as in the proof of the Theorem 1. The σ-mapping restricted on γ, denoted by σ_γ, is one to one and continuous, and is therefore a homeomorphism between γ and $\sigma(\gamma)$ on account of the compactness of γ. Let N be the interior of σ. Then $\sigma(N)$ is an open neighborhood of $\sigma(M_0)$ in T_0 and the closure of $\sigma(N)$ in T_0 is $\sigma(\gamma)$ which is compact. The local compactness of T_0 is proved.

Theorem 3. A complete commutative semi-normed *-algebra A with a family $\mathcal{V}$ of semi-norms is equivalent to the algebra $C(T, K)$ with k-topology, of all complex continuous functions on a locally compact Hausdorff space T if and only if for each $V_0\in\mathcal{V}$ there is $x_0\in A$ such that $\sup_{M\in\mathcal{M}}|x_0(M)|=2$, $x_0(M_0)=0$ for some closed maximal ideal M_0 belonging to the support of V_0, and $\bar{V}=\sup\{V: V(x_0)\leq 1, V\in\mathcal{V}\}$ is a seminorm in $\mathcal{V}$.

Proof. *Necessity.* By Theorem 2, there is $x_0\in M_0$ to each closed maximal ideal M_0 in A such that all the closed maximal ideals satisfying $|x_0(M)|\leq 2$ contain a kernel ideal E of some seminorm V_0. $\bar{V}=\sup\{V: V(x_0)\leq 1, V\in\mathcal{V}\}$ satisfies the relation $\bar{V}(x)=\sup\{|x(M)|: |x_0(M)|\leq 1, M\in\bar{Q}_{V_0}\}$ for all $x\in A$. Then the compact set $\{M: |x_0(M)| \leq 1, M\in\bar{Q}_{V_0}\}$ in T_0 is the support of $\bar{V}$ and $\bar{V}$ is a seminorm in $\mathcal{V}$.

Sufficiency. Let M_0 be a closed maximal ideal in A and Z_{V_0} a kernel ideal of some seminorm V_0 contained in M_0. We denote the set of closed maximal ideals M in A a satisfying $|x_0(M)|\leq\varepsilon<\frac{1}{2}$ for the $x_0\in M_0$ by W. Each $M_1\in W$ contains a kernel ideal Z_{V_1}. $G=\{M: x_0(M) \leq h<1, h>\frac{1}{2}, M\in\bar{Q}_{V_0}\}$ is a compact set in $\bar{Q}_{V_0}$ and is a support of some seminorm V'. $Z_{V'}$ is contained in M_1 since M_1 belongs to G. $V'(x_0)<1$ implies $\bar{V}\geq V'$ and $Z_{\bar{V}}\subset Z_{V'}$. Then $Z_{\bar{V}}\subset M_1$. Hence all the closed maximal ideals in E contain $Z_{\bar{V}}$ and the local compactness of T_0 follows from Theorem 2.

Let $\mathcal{L}$ be the set of all closed, symmetric, m-convex, and radial at 0 neighborhoods of LMC algebra A. It is clear that the closed m-convex set associated with $\bar{V}$ in Theorem 3 is given by $C_{\bar{V}}=\bigcap_{\substack{C_V\in\mathcal{L}\\ x_0\in C_V}} C_V$. Theorem 3 can be put into the following equivalent form.

Theorem 3′. A complete commutative *-algebra A with a family $\mathcal{V}$ of seminorms is equivalent to the algebra $C(T_0, K)$, with k-topology, of all complex continuous functions on a locally compact Hausdorff space T_0 if and only if, to each $M_0\in\mathcal{M}(A)$, there is an $x_0\in M_0$ such that (1) $\sup_{M\in\mathcal{M}}|x_0(M)|=2$ and (2) C_V is a neighborhood of 0.

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