143. *On Almost Everywhere Convergence of Walsh-Fourier Series**)

By Jun TATEOKA

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1968)

1. Introduction. L. Carleson [2] proved that the Fourier series of functions belonging to the class $L^2(-\pi, \pi)$ converge almost everywhere.

Combining the method of Carleson and the theory of interpolation of operators, R. A. Hunt [3] extended the result to the Fourier series of functions $f \in L^p(-\pi, \pi)$, $p>1$. In fact he proved three maximal theorems about partial sum of Fourier series. On the other hand P. Billard [1] applied the method of Carleson to Walsh-Fourier series of functions $f \in L^2(0, 1)$.

In the present paper, the author applies the Carleson-Hunt-Billard method to Walsh-Fourier series, and proves the analogues to Hunt's result.

Let $S_n(f)$ be the n-th partial sum of Walsh-Fourier series of integrable and periodic function $f(t)$ $(0\leq t\leq 1)$.

Let

$$Mf(t)=\mathrm{Sup}\,\{|S_n(f)| : n\geq 0\},$$

then the theorems of this paper are;

Theorem 1. *If* $1<p<\infty$, *then* $||Mf||_p\leq C_p||f||_p$.

Theorem 2. $||Mf||_1\leq C\int_0^1 |f(t)|(\log|f(t)|)^2dt+C.$

Theorem 3. *For any* $y>0$,

$$m\{t \in (0, 1)\,;\, Mf(t)>y\}\leq C \exp\{-Cy/||f||_\infty\}.$$

It is well known that these results imply the almost everywhere convergence of $S_n(f)$ to f(t) for f in the respective function spaces.

2. Notation. Let $(r_1, r_2, \cdots, r_n, \cdots)$ and $(w_0, w_1, \cdots, w_n, \cdots)$ be the classical system of Rademacher and Walsh functions. For a positive integer n we define N_n by $2^{N_n}\leq n<2^{N_n-1}$ and write

(2.1) $\quad n=\zeta_0+\zeta_1 2^1+\cdots+\zeta_{N_n}2^{N_n}(\zeta_j=0, 1\,;\, j=0, 1, 2, \cdots, N_n\,;\, \zeta_{N_n}=1).$

The Dirichlet kernel of Walsh system is defined by

(2.2) $$W_n(t)=w_0(t)+w_1(t)+\cdots+w_{n-1}(t).$$

*) This work was done during the author's stay at Mathematical Institute, Tohoku University. The author thanks Professors G. Sunouchi, C. Watari, and S. Igari for guidance and encouragement. Professor Sunouchi says that the analogous theorems have been established also by Hunt-Taiblesson from a letter of Hunt. But this work was done independently, and completed before the end of March, 1968.

We assume $\zeta_{N_n}=\zeta_{n_1}=\cdots=\zeta_{n_k}=1$ $(N_n>n_1>\cdots>n_k\geq 0)$ in (2.1).

Then (2.2) becomes

$$W_n(t)=\prod_{j=1}^{N_n}(1+r_j(t))+r_{N_n+1}\prod_{j=1}^{n_1}(1+r_j(t))+\cdots+r_{N_n+1}\cdots r_{n_{k-1}+1}\times\prod_{j=1}^{n_k}(1+r_j(t)),$$

where $\prod_{j=1}^{n_k}(1+r_j(t))=1$ for $n_k=0$.

Furthermore we can write

$$W_n(t)=w_n(t)[\delta_{N_n}^*(t)+\delta_{n_1}^*(t)+\cdots+\delta_{n_k}^*(t)], \tag{2.3}$$

where $\delta_j^*(t)=\begin{cases} 2^j & \text{for } 0<t<2^{-j-1}, \\ -2^j & \text{for } 2^{-j-1}<t<2^{-j} \ (j=0,1,2,\cdots), \\ 0 & \text{otherwise.}\end{cases}$

Observe

$$\delta_j^*(t)=\sum_{\nu=2^j}^{2^{j+1}-1}w_\nu(t)=r_{j+1}\sum_{\nu=0}^{2^j-1}w_\nu(t)=r_{j+1}(t)\prod_{\nu=1}^{j}(1+r_\nu(t)). \tag{2.4}$$

We put

$$\delta_n(t)=\delta_{N_n}^*(t)+\delta_{n_1}^*(t)+\cdots+\delta_{n_k}^*(t).$$

Let us write

$$S(f)\sim\sum_{n=0}^{\infty}c_nw_n(t),\quad \left(c_n=\int_0^1 f(t)w_n(t)dt,\ n=0,1,2,\cdots\right),$$

and put

$$\Delta_k(t)=\Delta_k(f;t)=\sum_{n=2^k}^{2^{k+1}-1}c_nw_n(t),\quad (k=0,1,2,\cdots),$$

$$\Delta_{-1}(t)=\Delta_{-1}(t)=c_0.$$

Then we get

$$S(f)\sim\sum_{k=-1}^{\infty}\Delta_k(t). \tag{2.5}$$

For each integer $\nu\geq 0$ we divide $(-2,2)$ into $4\cdot 2^\nu$ equal intervals of length $2^{-\nu}$. The resulting intervals are from left to right denoted $\omega_{j\cdot\nu}$, $j=-2\cdot 2^\nu,\cdots,2\cdot 2^\nu-1$. If we are not interested in subindex j or $j\cdot\nu$, we may write ω_ν or ω instead of $\omega_{j\cdot\nu}$.

For each integer $n>0$ we define $n[\omega_\nu]$ by the greatest integer less than or equal $2^{-\nu}n$.

We consider the modification of usual $(0,1)$ which is the set $(0,1)^*$ of points $t=(\xi_1,\xi_2,\cdots,\xi_n,\cdots)$ $(\xi_i=0,1)$ and make it totally disconnected compact abelian group. Let ω_ν be the set of points $t=(\xi_1^0,\xi_2^0,\cdots,\xi_\nu^0,\xi_{\nu+1},\cdots)$ in which $\xi_1^0,\xi_2^0,\cdots,\xi_\nu^0$ are fixed and $\xi_{\nu+1},\cdots$ vary independently. We transpose the structure of ω_ν to $(0,1)^*$ by the function $\varphi_{\omega_\nu}:\omega_\nu\to(0,1)^*$ defined by $\varphi_{\omega_\nu}[(\xi_1^0,\xi_2^0,\cdots,\xi_\nu^0,\xi_{\nu+1},\cdots)]=(\xi_{\nu+1},\xi_{\nu+2},\cdots)$. Then the n-th Walsh function on ω_ν $(\nu\geq 0)$ is $w_n(\omega_\nu;t)=w_n[\varphi_{\omega_\nu}(t)]$. In the same way we define the analogous function $\delta_n^*(\omega_\nu;t)$ and $\delta_n(\omega_\nu;t)$. P. Billard [1] verifies that

$w_n(t)=\theta w_{n[\omega_\nu]}(\omega_\nu;t)$ $(t\in\omega_\nu)$, where $\theta=\pm 1$ does not depend on t, and writing

$$S_n((0,1)^*;x)=\sum_{j=1}^{n-1}c_jw_j(x)=\int_0^1 f(t)w_n(x-t)\delta_n(x-t)dt$$

considers the modified partial sum of $S_n((0,1)^*;x)$;

$$S_n^*((0,1)^*;x)=\int_0^1 f(t)w_n(t)\delta_n(x-t)dt.$$

He observes

$$|S_n^*((0,1)^*;x)|=|S_n((0,1)^*;x)|.$$

3. Sketch of the proof. From the reduction of Hunt's theorem the following result implies the theorems.

Fix measurable set $F\subset(0,1)^*$ and consider the periodic function $f(x)=\chi_F(x)$, $x\in(0,1)^*$, and the number $1<p<\infty$ and $y>0$. For any fixed number $N>0$ we will show that for $|n|\leq \Lambda 2^{N-2}$ ($0<\Lambda<1$ is an absolute constant) and $x\in(0,1)^*$ we have $|S_n^*(x;\chi_F)|\leq$Const. Ly except for x in an exceptional set E, where $mE\leq$Const. $y^{-p}mF$, $L=L(p)\leq$Const. $p^3(p-1)^{-2}$.

We will study some of elements which are used in the proof of our result.

The following lemma is proved by C. Watari [4].

Lemma (3.1). *Let $f(t)$ be function of L^p class and its Walsh-Fourier series be formed* (2.5). *Then the series*

$$\sum_{k=-1}^{\infty}\eta_k\Delta_k(t)\qquad(\eta_k=0,1\ \textit{or}\ -1)$$

is Walsh-Fourier series of $g(t)$ of L^p class and there exists a constant A_p such that

$\|g(t)\|_p\leq A_p\|f(t)\|_p$, where $A_p\leq$Const. $p^2(p-1)^{-1}$, $1<p<\infty$.

We consider a suitable partition $\Omega=\{\omega_l\}$, $\omega_l=\omega_{\nu_l}$ of $(0,1)^*$. If $x\in\omega_l=\omega_l(x)$, we write

$$S_n^*((0,1)^*;x)=\theta\frac{1}{|\omega_l(x)|}\int_{\omega_l(x)}f(t)w_{n[\omega_l(x)]}(\omega_l(x);t)\delta_{n[\omega_l(x)]}(\omega_l(x);x-t)dt+R_n(x)+H_n(x),\tag{3.2}$$

where $\theta=\pm1$, $\delta_0=0$, $S_0=S_0^*=0$

$$\begin{cases} R_n(x)=\int_0^1 E_n(t)\delta_{n-n[\omega_l(x)]2^{\nu_l}}(x-t)dt,\\ H_n(x)=\int_0^1 [f(t)w_n(t)-E_n(t)]\delta_{n-n[\omega_l(x)]2^{\nu_l}}(x-t)dt,\\ E_n(t)=\frac{1}{|\omega_l(x)|}\int_{\omega_l(x)}f(u)w_n(u)du,\ t\in\omega_l(x).\end{cases}\tag{3.3}$$

From (2.1), (2.3), and (2.4) $R_n(x)$ is the finite sum of

$$\sum_{j=0}^{Nn}\zeta_j\Delta_j(E_n;t)\tag{3.4}$$

at the point x.

Lemma (3.5). *If $f(t)$ is integrable and $Tf(t)=\operatorname{Sup}_n |\sum_{j=-1}^{n} \Delta_j(f;t)|$, then*

$$\|Tf(t)\|_p \leq B_p \|f(t)\|_p, \text{ where } B_p \leq \text{Const. } p(p-1)^{-1},\ 1<p<\infty.$$

According to Lemmas (3.1) and (3.5) we have

$$\|R_n(x)\|_p \leq D_p \|E_n(x)\|_p, \text{ where } D_p \leq \text{Const. } p^3(p-1)^{-2},\ 1<p<\infty.$$

Then the extrapolation the theorem yields

(3.6) $m\{X \in \omega\,;\, R_n(x)>y\} \leq \text{Const. } \exp\{-\text{Const. } y/\|E_n(x)\|_\infty\}|\omega|.$

Basing ourselves on (3.6) and a slight modification of Hunt-Billard's result we can prove theorems.

References

[1] P. Billard: Sur la convergence presque partout des séries de Fourier-Walsh des fonctions de l'espace $L^2(0, 1)$. Studia Math., **28**, 363–388 (1967).

[2] L. Carleson: On convergence and growth of partial sums of Fourier series. Acta Math., **116**, 135–157 (1966).

[3] R. A. Hunt: On the convergence of Fourier series (to appear).

[4] C. Watari: Mean convergence of Walsh Fourier series. Tohoku Math. J., **16**, 183–188 (1964).