137. *Characteristic Classes for Spherical Fiber Spaces*[1)]

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1. Statement of results. Let $SF=SG=\lim_n SG(n)$, $SG(n)=\{f:S^n \to \text{degree } 1\}$, B_{SF} be the classifying space of SF. Our purpose is to determine $H_*(B_{SF}, Z_p)$ as a Hopf-algebra over Z_p, where p is an odd prime number. Coefficient is always Z_p, and we omit it in the sequel. Let $Q_0(S^0)=\lim_{\overrightarrow{n}} \Omega_0^n S^n$. Then $Q_0(S^0)$ has the same homotopy type of SF. Let $i: Q_0(S^0)\to SF$ be the homotopy equivalence. Dyer-Lashof determined $H_*(Q_0(S^0))$ as an algebra over Z_p. $H_*(Q_0(S^0))$ is a free commutative algebra generated by $x_J, J\in H$, where $H=\{J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)\}$ satisfies the following properties: 1) $r\geq 1$, 2) $j_i\equiv 0, (p-1)$, 3) $j_r\equiv 0$ $(2(p-1))$, 4) $(p-1)\leq j_1\leq j_2 \cdots \leq j_r$ 5), $\varepsilon_i=0$ or 1, 6) if $\varepsilon_{i+1}=0$ then $j_i/(p-1)$ and $j_{i+1}/(p-1)$ are even parity, if $\varepsilon_{i+1}=1$ then $j_i/(p-1)$ and $j_{i+1}/(p-1)$ are odd parity. There is a continuous map $h_0: L_p\to Q_0(S^0)$, and $x_j\equiv h_{0*}(e_{2j(p-1)})$, where $e_i\in H_i(L_p)$ is a generator, and $x_I \equiv x_{(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)}\equiv \beta_p^{\varepsilon_1}Q_{j_1}\cdots \beta_p^{\varepsilon_r -1}Q_{j_{r-1}}\beta_p^{\varepsilon_r}x_{j_r}$, where Q_j is the extended power operation defined by Dyer-Lashof. We identify $H_*(Q_0(S^0))$ and $H_*(SF)$ by i_* as a Z_p-module and we denote $\tilde{x}=i_*(x)$, if $x\in H_*(Q_0(S^0))$.

Theorem 1. *$H_*(SF)$ is a free commutative algebra generated by $\tilde{x}_J$; $J\in H$. Even though i_* is not a ring homomorphism.*

Let H_1 be the subset of H consisting of $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, such that $j_1\neq p-1$, and $r\geq 2$. Let $H_2=\{(\varepsilon, p-1, 1, j)\}\subseteq H$. And let $H_i^+ =\{J\in H_i, \deg(x_J)=\text{even}\}$, $H_i^-=\{J\in H_i, \deg(x_J)=\text{odd}\}$ $i=1, 2, \cdots$. Let $j; B_{SO}\to B_{SF}$ be the inclusion map. Then by Peterson-Toda, $H_*(B_{SO})/\ker j^*\cong Z_p[z_1, z_2, \cdots]$, where $\deg(z_j)=2j(p-1)$, and $\Delta(z_j)=\sum_{j_1+j_2=j} z_{j_1}\otimes z_{j_2}$, $z_0=1$. Let $\tilde{z}_j=j_*(Z_j)\in H(B_{SF})$.

Theorem 2. *$H_*(B_{SF})=Z_p[\tilde{z}_1, \tilde{z}_2, \cdots]\otimes \Lambda(\sigma\tilde{x}_1, \sigma\tilde{x}_2, \cdots)\otimes C_*$. C_* is a free commutative algebra generated by $\tilde{x}_J, J\in H_1\cup H_2$. $\sigma; H_*(SF)\to H_*(B_{SF})$ is suspension. $\sigma\tilde{x}_j, \sigma\tilde{x}_J$ are primitive elements, and $\Delta(\tilde{z}_j) = \sum_{j_1+j_2=j}\tilde{z}_{j_1}\otimes\tilde{z}_{j_2}$.*

$H^*(B_{SF})=Z_p[q_1, q_2, \cdots]\otimes\Lambda(\Delta q_1, \Delta q_2, \cdots)\otimes C$. $C=\bigotimes_{I\in H_1^+\cup H_2^+}\Lambda((\sigma\tilde{x}_I)^*) \otimes \bigotimes_{J\in H_1^-\cup H_2^+}\Gamma_p[(\sigma\tilde{x}_J)^*]$, where $(\ )^*$ denotes dual elements, where q_j is the j-th Wu-class, $j=1, 2, \cdots$.

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2. H-structures on $Q_0(S^0)$. Let $SF(n)=\{f:(S^n,*)\to(S^n,*)$, degree $1\}$. Then, $SG(n)$, and $SF(n)$ become H-spaces by composition of maps. Let $SF(n)\times SF(n)\xrightarrow{\wedge}SF(2n)$, $SG(n)\times SG(n)\xrightarrow{*}SG(2n)$ be the map defined by reduced join and join respectively, then these three maps, $\cdot$, $\wedge$, $*$, are homotopic in the stable range. Let $i_n:\Omega_0^nS^n\to\Omega_1^nS^n=SF(n)$ be the map defined by $i_n(l)=(i_n\vee l)$, and $i:Q_0S^0\to SF$ be the limit of i_n.

Proposition 2-1. The following diagram is homotopy commutative.

$$\begin{array}{ccccccc}
Q_0S^0\times Q_0S^0 & \xrightarrow[i\times i]{} & SF\times SF & \xrightarrow[\wedge]{} & SF & \xleftarrow[i]{} & Q_0S^0 \\
\downarrow{\scriptstyle \Delta\times\Delta} & & & & & & \uparrow{\scriptstyle \vee} \\
(Q_0S^0\times Q_0S^0)\times(Q_0S^0\times Q_0S^0) & & \xrightarrow[id\times T\times id]{} & & (Q_0S^0\times Q_0S^0)\times(Q_0S^0\times Q_0S^0) & \xrightarrow[\vee\times\wedge]{} & Q_0S^0\times Q^0S^0
\end{array}$$

where $\vee: Q_0S^0\times Q_0S^0$ be loop multiplication, and $\wedge: Q_0S^0\times Q_0S^0\to Q_0S^0$ be the map defined by reduced join.

If K is a CW-complex, we put $Q(K)=\lim\limits_{\to}\Omega^nS^nK$. $\theta: W\times_{\pi_p}Q(K)^p\to Q(K)$ be the map defined by Dyer-Lashof. Let $Q(K)\times Q(L)\to Q(K\wedge L)$ be the map defined by reduced join.

Proposition 2-2. The following diagram is homotopy commutative.

$$\begin{array}{ccccc}
Q(K)\times(W\times_{\pi_p}Q(L)^p) & \xrightarrow[id\times\theta]{} & Q(K)\times Q(L) & \xrightarrow[\wedge]{} & Q(K\wedge L) \\
\downarrow & & & & \uparrow{\scriptstyle \theta} \\
W\times_{\pi_p}(Q(K)\times Q(L)^p) & \xrightarrow[id\times(\Delta_p\times id)]{} & W\times_{\pi_p}(Q(K)\times Q(L))^p & \xrightarrow[id\times_{\pi_p}(\wedge)^p]{} & W\times_{\pi_p}Q(K\wedge L)^p
\end{array}$$

Let $h:L_p=W/\pi_p\to Q(S^0)=\lim\limits_{\underset{n}{\to}}\Omega^nS^n$ be the map defined by $h:L_p=W/\pi_p\to W\times_{\pi_p}(w)^p\to W\times_{\pi_p}Q(S^0)\xrightarrow{\theta}Q(S^0)$, $w\in Q_1(S^0)$, $h_0:L_p\xrightarrow{h}Q_p(S^0)\xrightarrow{(-p\cdot id)}Q_0(S^0)$.

Proposition 2-3. The following diagram is homotopy commutative.

$$\begin{array}{ccccc}
Q(K)\times L_p & \xrightarrow{id\times h} & Q(K)\times Q(S^0) & \xrightarrow{\wedge} & Q(K\wedge S^0) \\
\downarrow{\scriptstyle T} & & & & \uparrow{\scriptstyle \cong} \\
L_p\times Q(K) & \xrightarrow[id\times_{\pi_p}\Delta_p]{} & W\times_{\pi_p}Q(K)^p & \xrightarrow[\theta]{} & Q(K)
\end{array}$$

3. Proof of Theorem 1. We introduce a filtration into $H_*(Q_0(S^0))$. $H_*(Q_0(S^0))=G_0\supseteq G_1\supseteq G_2\cdots$ satisfies the following properties. 1) $G_1=\ker\varepsilon$, $\varepsilon: H_*(Q_0(S^0))\to Z_p$ is the augumentation. 2) $G_i\otimes G_j$

$\to G_{i+j}$, 3) $x_J \in G_{p^r-1}$ where $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r) \in H$, and $x_J \notin G_{p^r-1+1}$.

Proposition 3-1. There exists unique filtration in $H_*(Q_0(S^0))$ satisfying the properties 1), 2), 3), and for $x \in H_*(Q_0(S^0))$, if $x \in G_j$ and $\Delta x = 1\otimes x + x\otimes 1 + \Sigma x'\otimes x''$, then x', x'' belong G_j.

Proposition 3-2. Let $E_0H_*(Q_0(S^0))$ be the algebra associated to the above filtration. Then $H_*(Q_0(S^0))$ and $E_0H_*(Q_0(S^0))$ are isomorphic as algebras.

Proposition 3-3. $\wedge_*(x\otimes y) \in G_{pij}$, if $x \in G_i$ and $y \in G_j$.

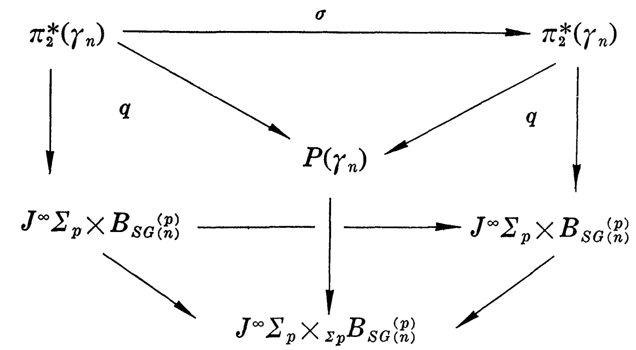
Then Theorem 1 follows from Propositions 2-1, 3-1, 3-2, and 3-3.

4. H_p^∞-structure on B_{SF}. Let $\gamma_n \to B_{SG(n)}$ be the universal oriented spherical fiber space with fiber S^{n-1}. Σ_p denotes the permutation group of p-element. $J^m\Sigma_p = \Sigma_p * \cdots * \Sigma_p$ denote m-th join of Σ_p. Let $\gamma_n^{(p)} \to B_{SG(n)}^{(p)}$ be exterior p-th Whiteney join of γ_n. Let $\pi_2^*(\gamma_n) \to J^m\Sigma_p \times B_{SG(n)}^{(p)}$ denote the induced fibering of $\gamma_n^{(p)}$ by $\pi_2 : J^m\Sigma_p \times B_{SG(n)}^{(p)} \to B_{SG(n)}^{(p)}$.

Proposition 4-1. There exists a spherical fibering $P(\gamma_n) \to J^\infty\Sigma_p \times_{\Sigma_p} B_{SG(n)}^{(p)}$ with fiber S^{pn-1}, and bundle map

$$\begin{array}{ccc} q : \pi_2^*(\gamma_n) & \to & P(\gamma_n) \\ \downarrow & & \downarrow \\ J^\infty\Sigma_p \times B_{SG(n)}^{(p)} & \to & J^\infty\Sigma_p \times_{\Sigma_p} B_{SG(n)}^{(p)} \end{array}$$

They satisfy following commutative diagram. $\forall \sigma \in \Sigma_p$



Let $E_{SG(n)} \to B_{SG(n)}$ be the principal fibering associated with γ_n, i.e.

$$\begin{array}{l} E_{SG(n)} = \{f : S^{n-1} \to \gamma_n \text{ ; oriented fiber map}\}. \\ \qquad\qquad\quad \downarrow \qquad\quad \downarrow \\ \qquad\qquad\quad * \to B_{SG(n)} \end{array}$$

$P_0(\gamma_n) \to J^m\pi_p \times_{\pi_p} B_{SG(n)}^{(p)}$ denotes restricted fibering of $P(\gamma_n)$, where π_p denotes cyclic group of order p. $\bar{\theta}$; $J^m\pi_p \times_{\pi_p} B_{SG(n)}^{(p)} \to B_{SG(pn)}$ be the classifying map of $P_0(\gamma_n)$. As the map $\bar{\theta} : J^m\pi_{p/\pi_p} \to J^m\pi_p \times_{\pi_p} (e_0)^p \to J^m\pi_p \times_{\pi_p} B_{SG(n)}^{(p)} \xrightarrow{\bar{\theta}} B_{SG(pn)}$, $e_0 \in B_{SG(n)}$, is induced by the n-times of the regular representation: $\pi_p \to SO(pn) \to SG(pn)$, by the result of Kambe, we may suppose the above map is homotopic to constant map for suitable m and n. And we may take m, sufficiently large for a suitablly sufficient large n. So we may assume $\bar{\theta}(J^m\pi_{p/\pi_p}) = e_0 \in B_{SG(pn)}$.

We define a map $\bar{\rho}: J^m\pi_p \to SG(pn)$ in the following way. We identify $SG(n)=(E_{SG(n)})_{e_0}=\pi^{-1}(e_0)$, and $SG(pn)=(E_{SG(pn)})=\pi^{-1}(e_0)$, respectively. We fix $i_n \in (E_{SG(n)})_{e_0}$ and for $w \in J^m\pi_p$, $\bar{\rho}(w)$ represents the following map.

$$\begin{array}{ccccccc}
\bar{\rho}(w): S^{pn-1} & \xrightarrow[(w,\, id_n * \cdots * id_n)]{} & J^m\pi_p \times \gamma_n^{(p)} = \pi_2^*(\gamma_n) & \xrightarrow[q]{} & P_0(\gamma_n) & \longrightarrow & \gamma_{pn} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & J^m\pi_p \times B_{SG(n)}^{(p)} & \longrightarrow & J^m\pi_p \times_{\pi_p} B_{SG(n)}^{(p)} & \longrightarrow & B_{SG(pn)}
\end{array}$$

We define $\bar{\theta}': J^m\pi_p \times E_{SG(n)}^{(p)} \to E_{SG(pn)}$ by the following commutative diagram, for $(w, f_1, \cdots, f_p) \in J^m\pi_p \times E_{SG(n)}^{(p)}$, $\bar{\theta}'(w, f_1, \cdots, f_p)$:

$$\begin{array}{ccccccccc}
S^{pn-1} & \xrightarrow[\bar{\rho}^{-1}(w)]{} & S^{pn-1} & \xrightarrow[(w,\, f_1 * \cdots * f_p)]{} & J^m\pi_p \times \gamma_n^{(p)} & \xrightarrow[q]{} & p_0(\gamma_n) & \longrightarrow & \gamma_{pn} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & * & \longrightarrow & J^m\pi_p \times B_{SG(n)}^{(p)} & \longrightarrow & J^m\pi_p \times_{\pi_p} B_{SG(n)}^{(p)} & \longrightarrow & B_{SG(pn)}
\end{array}$$

Proposition 4.2. $\bar{\theta}'$ is π_p-equivariant, we obtain following commutative diagram.

$$\begin{array}{ccc}
\bar{\theta}: J^m\pi_p \times_{\pi_p} E_{SG(n)}^{(p)} & \longrightarrow & E_{SG(pn)} \\
\downarrow & & \downarrow \\
J^m\pi_p \times_{\pi_p} B_{SG(n)}^{(p)} & \longrightarrow & B_{SG(pn)}
\end{array}$$

And $\bar{\theta}(J^m\pi_p \times_{\pi_p} SG(n)^{(p)}) \subseteq SG(pn) \subseteq E_{SG(pn)}$, and $\bar{\theta}(w, f_1, \cdots, f_p)=\bar{\rho}(w)(f_1 * \cdots * f_p)\bar{\rho}(w)^{-1}$, for any $(w, f_1, \cdots, f_p) \in J^m\pi_p \times_{\pi_p} SG(n)^{(p)}$.

5. Decomposition of $\bar{\theta}$. Let $\Lambda=\{J=(\varepsilon_1, \cdots, \varepsilon_p), \varepsilon_i=0 \text{ or } 1\}$, $|J|=$number of $\{\varepsilon_i=1, J=(\varepsilon_1, \cdots, \varepsilon_p)\}$. π_p operates on Λ by permutation. We introduce in Λ an total ordering by the lexicographic order, for example, $(0, 1, \cdots) \lneqq (1, \cdots)$. Let $\bar{\Lambda}=\Lambda/\pi_p$. We define the map $\bar{\Lambda} \xrightarrow{\pi} \Lambda$, by $\pi(\{J\})=$the first element in $\{J\}$. Λ_0 denotes the image of π. For each element $J_0 \in \Lambda_0$, we define $\eta_{J_0}: (\Omega_0^{n-1}S^{n-1})^p \to G(pn)$ as follows, where $G(pn)=\{f: S^{pn-1} \to S^{pn-1}\}$, $\varphi_2: S^{n-1} \to S_0^{n-1} \vee S_1^{n-1}$. For $(l_1, \cdots, l_p) \in (\Omega_0^{n-1}S^{n-1})^p$, $\eta_{J_0}(l_1, \cdots, l_p)$ represents following map.

$$\begin{array}{ccc}
\eta_{J_0}(l_1, \cdots, l_p): S^{n-1} * \cdots * S^{n-1} & \xrightarrow[\varphi_2 * \cdots * \varphi_2]{} & (S_0^{n-1} \vee S_1^{n-1}) * \cdots * (S_0^{n-1} \vee S_1^{n-1}) \\
\downarrow & & \downarrow \\
S^{pn-1} & \xleftarrow[\circledast]{} & \bigvee_{J \in \Lambda}^{\Lambda p-1} S_J^{pn-1}
\end{array}$$

$\circledast$ is the map as follows, $\circledast|_{S_J}: S_J \to S$ represents, a) $0 * \cdots * 0$, if $J \neq \sigma J_0$ for any $\sigma \in \pi_p$, $0: S^{n-1} \to S^{n-1}$. b) $l_1^{\varepsilon_1} * \cdots * l_p^{\varepsilon_p}$, if $J=\sigma J_0=(\varepsilon_1, \cdots, \varepsilon_p)$ for some $\sigma \in \pi$, where $l_i^0=id$, $l_i^1=l_i$. And $S_J^{pn-1}=S_{\varepsilon_1}^{n-1} * \cdots * S_{\varepsilon_p}^{n-1}$.

We define $\bar{\theta}'_{J_0}: J^m\pi_p \times (\Omega_0^{n-1}S^{n-1})^p \to G(pn)$, for each $J_0 \in \Lambda_0$, as $\bar{\theta}'_{J_0}(w, l_1, \cdots, l_p) = \bar{\rho}(w)\eta_{J_0}(l_1, \cdots, l_p)\bar{\rho}(w)^{-1}$.

Proposition 5.1. $\bar{\theta}'_{J_0}: J^m_{\pi_p} \times (\Omega_0^{n-1}S^{n-1})^p \to G(pn)$, is π_p-equivariant, therefore it defines the following map $\bar{\theta}_{J_0}: J^m\pi_p \times_{\pi_p} (\Omega_0^{n-1}S^{n-1})^p \to G(pn)$.

Let $i: G(pn) \to \Omega^{pn+1}S^{pn+1}$ be the inclusion.

Proposition 5.2. $i\bar{\theta}$ and $\bigvee_{J_0 \in \Lambda_0} i\bar{\theta}_{J_0}$ are homotopic on $(pn-5)$-skeleton as a map $J^m\pi_p \times_{\pi_p} (\Omega_0^{n-1}S^{n-1})^p \to \Omega^{pn+1}S^{pn+1}$, where $\vee$ denotes loop multiplication on $\Omega^{pn+1}S^{pn+1}$.

For $J_0 \in \Lambda_0$, $|J_0| \neq 0, p$, we define $h_{J_0}: J^m\pi_{p/\pi_p} \to G(pn)$ as follows, for $w \in J^m\pi_p$,

$$\begin{array}{ccccc} h_{J_0}(w): S^{pn-1} & \xrightarrow{\bar{\rho}(w)^{-1}} & S^{pn-1} & \xrightarrow{\varphi_2 * \cdots * \varphi_2} & \bigvee_{J \in \Lambda} S_J^{pn-1} \\ \downarrow & & & & \downarrow \circledast \\ S^{pn-1} & & \xleftarrow{\quad \bar{\rho}(w) \quad} & & S^{pn-1} \end{array}$$

where $\circledast |_{S_J}: S_J^{pn-1} \to S^{pn-1}$ represents a) $0 * \cdots * 0$, if $J \neq \sigma J_0$, for any $\sigma \in \pi_p$. b) id_{pn-1}, if $J = \sigma J_0$, for some $\sigma \in \pi_p$. h_{J_0} is well defined.

Proposition 5.3. For $\Lambda_0 \ni J_0 = (\varepsilon_1, \cdots, \varepsilon_p)$, $0 \lneqq |J| \lneqq p$, the following diagram is homotopy commutative.

$$\begin{array}{ccccc} J^m\pi_{p/\pi_p} \times \Omega_0^{n-1}S^{n-1} & \xrightarrow{id \times_{\pi_p} \Delta_p} & J^m\pi_p \times_{\pi_p} (\Omega_0^{n-1}S^{n-1})^{(p)} & \longrightarrow & G_0(pn) \\ {\scriptstyle h_{J_0} \times ( )^{\varepsilon_1} * \cdots * ( )^{\varepsilon_p}} \downarrow & & & & \downarrow {\scriptstyle * id_{pn-1}} \\ G(pn) \times G_0(pn) & & \xrightarrow{\quad * \quad} & & G_0(2pn) \end{array}$$

where $( )^{\varepsilon_1} * \cdots * ( )^{\varepsilon_p}: \Omega_0^{n-1}S^{n-1} \to G(pn)$ is the map defined by $l \to (l)^{\varepsilon_1} * \cdots * (l)^{\varepsilon_p}$.

We define $\tilde{\theta}_p: J^m\pi_p \times_{p\pi_p} (\Omega_0^{n-1}S^{n-1})^p \to G(pn)$ by $\tilde{\theta}_p(w, l_1, \cdots, l_p) = \bar{\rho}(w)(l_1 *, \cdots, * l_p)\bar{\rho}(w)^{-1}$.

Proposition 5.4. $\tilde{\theta}_p \simeq \bar{\theta}_{(1,\cdots,1)}$; homotopic.

6. Proof of Theorem 2. $\bar{\theta}: J^\infty\pi_p \times_{\pi_p} SF^p \to SF$, $\bar{\theta}: J^\infty\pi_p \times_{\pi_p} B^p_{SF} \to B_{SF}$ are the maps corresponding to $\bar{\theta}: J^m\pi_p \times_{\pi_p} SG(n)^p \to SG(pn)$, $\bar{\theta}: J^m\pi_p \times_{\pi_p} B^p_{SG(n)} \to B_{SG(pn)}$ for large m and n. We define $\bar{Q}_j: H_*(SF) \to H_*(SF)$, $\bar{Q}_j: H_*(B_{SF}) \to H_*(B_{SF})$, $j = 1, 2, \cdots$, by the $\bar{Q}_j(x) = \bar{\theta}_*(e_j \otimes_{\pi_p} x^p)$, for $x \in H_*(SF)$, or $\in H_*(B_{SF})$.

Proposition 6.1. In the homology spectral sequence associated with following fibering $SF \to E_{SF} \to B_{SF}$. $E^2_{**} = H_*(B_{SF}) \otimes H_*(SF)$. If $x \in E^2_{2n,0}$ is transgresive. $y \in E^2_{0,2n-1}$, $\tau(x) = \{y\}$, then we obtain the following relations. $\{\tau\bar{Q}_0(x)\} = \{\tau(x^p)\} = \{\bar{Q}_{p-1}(y)\}$ in $E^{2np}_{0,2np-1}$, and $\{\tau(x^{p-1} \otimes y)\} = \{\bar{Q}_{p-2}(y)\}$ in $E^{2n(p-1)}_{0,2np-2}$.

Proposition 6.2. If $\bar{x}_I \in H_*(SF)$ belongs to G_{pj}, $j \geq 1$, where $I \in H$, then $\bar{Q}_{p-2}(\tilde{x}_I)$, $\bar{Q}_{p-1}(\tilde{x}_I)$ belong to G_{pj+1}, and as elements of $G_{pj+1}/(G_{pj+1_{+1}}$ + decomp). they coincide with $\widetilde{\beta_p Q_{p-1}(x_I)}$, $\widetilde{Q_p(x_I)}$ respectively.

We consider $j_*: H_*(SO) \to H_*(SF)$, by Peterson-Toda, $H_*(SO)/\ker j_* = \Lambda(y_1, y_2, \cdots)$. $\deg(y_i) = 2i(p-1)-1$. Let $\tilde{y}_i \in H_*(SF)$, be $j_*(y_i)$.

Proposition 6-3. $H_*(SF)$ is a free commutative algebra generated by $\tilde{x}_j$, $\tilde{y}_j$, $j=1, 2, \cdots$, $\tilde{x}_I$, $I \in H_1^+ \cup H_2^+$, $\bar{Q}_{p-1} \cdots \bar{Q}_{p-1}(\tilde{x}_I)$, $I \in H_1^- \cup H_2^-$, $\bar{Q}_{p-1}$ operate on $\tilde{x}_I$ k-times, $k \geq 0$. $\bar{Q}_{p-2}\bar{Q}_{p-1} \cdots \bar{Q}_{p-1}(\tilde{x}_I)$, $I \in H_1^- \cup H_2^-$, $\bar{Q}_{p-2}$ operates on $\bar{Q}_{p-1} \cdots \bar{Q}_{p-1}(\tilde{x}_I)$ exactly one times, and $\bar{Q}_{p-1}$ operates on $\tilde{x}_I$, k-times, $k \geq 0$.

This proposition is proved by using prop, 6-2, and structure of $H_*(SF)$ as an algebra. Then Theorem 2 follows from Ppropositions 6-1, 6-3 and the comparision theorem for spectral sequence.

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