134. On Some Trigonometric Series

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1. Introduction and Theorems. 1.1. Let us consider a trigonometric series

$$(1)\qquad \frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

If we write $r_n=\sqrt{a_n^2+b_n^2}$, then the series (1) can be written in the form

$$(2)\qquad \sum_{n=0}^{\infty} r_n\cos(nx+c_n),\quad -\pi/2\leqq c_n<3\pi/2.$$

We denote by α_0 the root of the equation

$$\int_0^{3\pi/2} x^{-\alpha}\cos x\,dx=0.$$

It is known that $\alpha_0=0.30844\cdots$.

In the case $c_n=0$ $(n=1,2,\cdots)$ in (2), we have proved the following theorems [1], as generalization of Chowla's and Selberg's theorems [2].

Theorem I. *If the series* $\sum_{n=1}^{\infty} r_n/n^{1-\beta}$ *diverges for a* β, $0\leqq\beta<\alpha_0$, *then*

$$\limsup_{N\to\infty}\left\{-\min_{0\leqq x<\pi}\left(\sum_{n=1}^{N} r_n\cos nx\Big/\sum_{n=1}^{N} r_n\right)\right\}\geqq A(\beta)$$

where $A(\beta)=-(1-\beta)(2/(3\pi))^{1-\beta}\int_0^{3\pi/2}x^{-\beta}\cos x\,dx>0$. *The constant* $A(\beta)$ *is the best possible one.*

Theorem II. *Let* $0\leqq\beta<\alpha_0$. *If there exist* $A>0$ *and* λ, $0\leqq\lambda<1-\beta$ *such that*

$$\sum_{n=1}^{N} r_n\cos nx\geqq -AN^{\lambda}\quad \textit{for all } x \textit{ and all } N,$$

then the series $\sum_{n=1}^{\infty} r_n/n^{1-\beta}$ *converges.*

1.2. For non-vanishing (c_n), Chidambaraswamy and Shah [3] have proved the following generalization of Chowla's and Selberg's theorems [2].

Theorem III. *If* $r_0>0$,

$$\sum_{n=0}^{N} r_n\cos(nx+c_n)\geqq 0\quad \textit{for all } x \textit{ and all } N$$

and there exist $a>0$, $0<\beta<1$, *and* $d=d(a,\beta)>0$ *such that*

$$\sup_{n\geqq 1}\int_0^a x^{-\beta}\cos(x+c_n)dx\leqq -d,$$

then the series $\sum_{n=1}^{\infty} r_n/n^{1-\beta}$ *converges.*

Theorem IV. *Let* $0<\beta<1$, $1\leqq b<1/(1-\beta)$, $0<\gamma<1-b(1-\beta)$. *If*

$$\sup_{n\geqq 1}\int_0^{3\pi/2} x^{-\beta}\cos(x+c_n)dx\leqq -d<0$$

and the sequence of positive integers n_k $(k\geqq 1)$, *satisfies the condition*

$$1\leqq n_1<n_2<\cdots,\ n_N\leqq AN^{b+\varepsilon}\qquad (0<\varepsilon<1-(1-\beta)b-\gamma),$$

then

$$\limsup_{N\to\infty}\left\{-N^{-\gamma}\min_{0\leqq x\leqq 2\pi}\sum_{k=1}^{N}\cos(n_k x+c_k)\right\}>0.$$

1.3. We shall prove the theorems which contain above theorems as particular case.

Theorem 1. *If there exist* $a>0$, $0<\beta<1$, *and* $d>0$ *such that*

$$\sup_{n\geqq 1}\int_0^a x^{-\beta}\cos(x+c_n)dx=-d \tag{3}$$

and the series $\sum_{n=1}^{\infty} r_n/n^{1-\beta}$ *diverges, then*

$$\limsup_{N\to\infty}\left\{-\min_{0\leqq x\leqq \pi}\left(\sum_{n=1}^{N} r_n\cos(nx+c_n)\Big/\sum_{n=1}^{N} r_n\right)\right\}\geqq A'(\beta) \tag{4}$$

where $A'(\beta)=d(1-\beta)/a^{1-\beta}$. *The constant* $A'(\beta)$ *is the best possible one.*

If $c_n=0$ for all n in (2), then the theorem reduces to Theorem I. If all coefficients a_n and b_n in the series (1) are non-negative, then $0\leqq c_n\leqq \pi/2$ and then we can take $a=3\pi/2$ in (3) and (3) is satisfied by $\beta<\alpha_0$. Thus we get the following

Corollary 1. *If* $a_n\geqq 0$ *and* $b_n\geqq 0$ *for all* n *in* (1) *and the series* $\sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}/n^{1-\beta}$ *diverges for a* β, $0\leqq\beta<\alpha_0$, *then*

$$\limsup_{N\to\infty}\left\{-\min_{0\leqq x\leqq 2\pi}\left(\sum_{n=1}^{N}(a_n\cos nx+b_n\sin nx)\Big/\sum_{n=1}^{N}\sqrt{a_n^2+b_n^2}\right)\right\}\geqq A''(\beta)$$

where $A''(\beta)$ *is a positive constant.*

If one of c_n is $-\pi/2$ or (c_n) has limiting point $-\pi/2$, then the equation (3) does not hold for any a, any β, and any d. Suppose that $-\pi/2+\delta\leqq c_n\leqq(3\pi/2)-\delta$ for all n, then, taking $a=2\pi-\delta$, the equation (3) has a solution. If $c_n=\pi/2$, then the series (1) becomes the sine series $-\sum_{n=1}^{\infty} r_n\sin nx$ and the relation (3) holds for any β, $0<\beta<1$, and for any $a>0$, hence we have

Corollary 2. *In the case of sine series* $\sum_{n=1}^{\infty} b_n\sin nx$, *if* $b_n\geqq 0$ *for* *all n and the series* $\sum_{n=1}^{\infty} b_n/n^{\alpha}$ *diverges for an* α, $0<\alpha<1$, *then*

$$\limsup_{N\to\infty}\left\{\max_{0\leqq x\leqq \pi}\left(\sum_{n=1}^{N} b_n \sin nx\Big/\sum_{n=1}^{N} b_n\right)\right\}\geqq A'''$$

where A''' *is a positive constant.*

Finally suppose that (n_k) is an increasing sequence of integers and that $a_{n_k}=1$ for $k=1, 2, \cdots$ and the other a_n are zero. Then Theorem 1 reduces to

Corollary 3. *If there exist* $a>0$, $0<\beta<1$, *and* $d>0$ *such that*

$$\sup_{n\geqq 1}\int_0^a x^{-\beta}\cos(x+c_{n_k})dx=-d$$

and $\sum_{k=1}^{\infty} n_k^{\beta-1}$ *diverges, then*

$$\limsup_{N\to\infty}\left\{-\min_{0\leqq x\leqq 2\pi}\left(\frac{1}{N}\sum_{k=1}^{N}\cos(n_k x+c_{n_k})\right)\right\}\geqq A''''(\beta)$$

where $A''''(\beta)=d(1-\beta)a^{1-\beta}>0$.

This contains Theorem IV as a particular case.

Theorem 1 can be stated in the following equivalent form.

Theorem 1′. *If there exist* $a>0$, $0<\beta<1$, *and* $d>0$ *satisfying the condition* (3) *and if, for any* δ, $0<\delta<1$,

$$\sum_{n=1}^{N} r_n\cos(nx+c_n)\geqq -\delta A'(\beta)\sum_{n=1}^{N} r_n \quad \textit{for all } N \textit{ and all } x,$$

then the series $\sum_{n=1}^{\infty} r_n/n^{1-\beta}$ *converges.*

Theorem III *is a particular case of Theorem* 1′ *and then of Theorem* 1 *since we can suppose that* $\sum_{n=1}^{\infty} r_n$ *diverges.*

Our second theorem is as follows:

Theorem 2. *If there exist* $a>0$, $0<\beta<1$, *and* $d>0$, *satisfying the condition* (3) *and further if there exist* $A>0$ *and* $0<\lambda<1-\beta$ *such that*

$$\sum_{n=1}^{N} r_n\cos(nx+c_n)\geqq -AN^{\lambda} \quad \textit{for all } N \textit{ and } x, \tag{5}$$

then the series $\sum_{n=1}^{\infty} r_n/n^{1-\beta}$ *converges.*

If all c_n vanish, then this theorem reduces to Theorem II, and further Theorem III is also a particular case of Theorem 2. If $c_n=\pi/2$ for all n, then the relation (3) is satisfied for any β, $0<\beta<1$, and a suitable a and the left side of (5) reduces to the sine series $-\sum_{n=1}^{\infty} r_n \sin nx$. Thus we get

Corollary 4. *If there exist* $A>0$ *and* λ, $0<\lambda<a<1$, *such that*

$$\sum_{n=1}^{N} r_n \sin nx\leqq AN^{\lambda} \quad \textit{for all } N \textit{ and } x,$$

then the series $\sum_{n=1}^{\infty} r_n/n^{\alpha}$ *converges.*

2. Proof of Theorems. 2.2. Proof of Theorem 1. Consider $a>0$, $0<\beta<1$, and $d>0$, satisfying the condition (3), then

$$\begin{aligned} -d\sum_{n=1}^{N}\frac{r_n}{n^{1-\beta}} &\geqq \sum_{n=1}^{N}\frac{r_n}{n^{1-\beta}}\int_0^a x^{-\beta}\cos(x+c_n)dx \\ &= \sum_{n=1}^{N} r_n\int_0^{a/n} x^{-\beta}\cos(nx+c_n)dx = \sum_{n=1}^{N} r_n \sum_{j=n}^{\infty}\int_{a/(j+1)}^{a/j} \\ &= \sum_{n=1}^{N}\sum_{j=n}^{N} + \sum_{n=1}^{N}\sum_{j=n+1}^{\infty} = \sum_{j=1}^{N}\sum_{j=1}^{j} + \sum_{j=N+1}^{\infty}\sum_{n=1}^{N} \qquad (6) \\ &= \sum_{j=1}^{N-1}\int_{a/(j+1)}^{a/j}\left(\sum_{n=1}^{j} r_n\cos(nx+c_n)\right)x^{-\beta}\,dx \\ &\qquad + \int_0^{a/N}\left(\sum_{n=1}^{N} r_n\cos(nx+c_n)\right)x^{-\beta}\,dx. \end{aligned}$$

If we put $s_j=\sum_{n=1}^{j} r_n$ and suppose that

$$M(j)=\min_{0\leqq x\leqq 2\pi}\left(\sum_{n=1}^{j} r_n\cos(nx+c_n)\right)\geqq -\delta A'(\beta)s_j \qquad (7)$$

for all j, $1\leqq j\leqq N$, and for some δ, $0<\delta<1$, then (6) gives

$$\begin{aligned} -d\sum_{n=1}^{N}\frac{r_n}{n^{1-\beta}} &\geqq -\delta A'(\beta)\left(\sum_{j=1}^{N-1} s_j\int_{a/(j+1)}^{a/j} x^{-\beta}\,dx + s_N\int_0^{a/N} x^{-\beta}\,dx\right) \\ &= -\delta d\left(\sum_{j=1}^{N-1} s_j(j^{\beta-1}-(j+1)^{\beta-1}) + s_N N^{\beta-1}\right) \\ &= -\delta d\sum_{j=1}^{N}\frac{r_j}{j^{1-\beta}}. \end{aligned}$$

This is a contradiction and then, for any δ, $0<\delta<1$, there is a j such that the relation (7) does not hold. By divergence of the series $\sum_{n=1}^{\infty} r_n/n^{1-\beta}$, we have, for any δ, $0<\delta<1$,

$$M(j)<-\delta A'(\beta)s_j \quad \text{for infinitely many } j$$

and then

$$\limsup_{N\to\infty}(-M(N)/s_N)\geqq \delta A'(\beta).$$

Since δ is any positive number <1, we get the required relation (4).

2.2. Proof of Theorem 2. By (6) and the assumption (5),

$$\begin{aligned} -d\sum_{n=1}^{N}\frac{r_n}{n^{1-\beta}} &\geqq -A\sum_{j=1}^{N-1} j^{\lambda}\int_{a/(j+1)}^{a/j} x^{-\beta}dx - AN^{\lambda}\int_0^{a/N} x^{-\beta}\,dx \\ &\geqq -A\int_0^a x^{-\lambda-\beta}\,dx = -Aa^{1-\lambda-\beta}/(1-\lambda-\beta) \end{aligned}$$

and then

$$\sum_{n=1}^{N}\frac{r_n}{n^{1-\beta}}\leqq Aa^{1-\lambda-\beta}/(1-\lambda-\beta)d \quad \text{for all } N.$$

Thus we get the convergence of the series $\sum_{n=1}^{\infty} r_n/n^{1-\beta}$.

References

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