133. *Criteria for Oscillation of Solutions of Differential Equations of Arbitrary Order*[1)]

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H. Onose extending a result of the author [2], gave in [7] a sufficient condition for all solutions of the equation

$$(*) \qquad x^{(n)}+p(t)g(x, x', \cdots, x^{(n-1)})=0$$

to oscillate, provided that n is even and g homogeneous of degree $2s+1$.

Here we improve Onose's result considerably, by assuming quite weaker conditions which guarantee the oscillation of all solutions of $(*)$, and moreover, we consider the case $n=\text{odd}$. Thus, we also improve a result due to Howard ([1], Theorem 2), and generalize results of Ličko and Švec [5], and Mikusiński [6].

All functions considered are supposed to be continuous on their domains, and such that they guarantee the existence of solutions of $(*)$ for all large t (n will always be supposed to be >1). In what follows, we consider only such solutions which are nontrivial for all large t. By an oscillatory solution of $(*)$, we mean a solution with arbitrarily large zeros.

1. The following theorem has been proved in [4]:

Theorem 1. *For n even, let $(*)$ satisfy the following assumptions:*

(i) $\quad p:I\to \boldsymbol{R}_+=(0, +\infty), \quad I=[t_0, +\infty), \quad t_0\geq 0,$ *and*

(S) $$\int_{t_0}^{\infty} t^{n-1}p(t)dt=+\infty;$$

(ii) $$g:\boldsymbol{R}^n\to\boldsymbol{R}=(-\infty, +\infty), \quad x_1g(x_1, x_2, \cdots, x_n)>0$$
for every $(x_1, \cdots, x_n)\in \boldsymbol{R}^n$ *with* $x_1\neq 0$;

then every bounded solution of $(*)$ is oscillatory.

Now we show that an analogous result holds for the case $n=\text{odd}$. In fact, we establish the following

Lemma. *Suppose that n is odd, and that the functions p, g satisfy the hypotheses of Theorem 1; then every bounded solution of $(*)$ is oscillatory, or tends to zero monotonically as $t\to+\infty$.*

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Proof. Since $x^{(n)}(t)=-p(t)g(x(t), x'(t), \cdots, x^{(n-1)}(t))$, it follows that if $x(t)$ is positive and bounded for all large t, we must have $(-1)^k x^{(k)}(t)>0$, for every $k=1, 2, \cdots, n-1$, and every $t\geq$ (some fixed) $T\geq t_0$. In fact, due to the boundedness of $x(t)$, no two consecutive derivatives of $x(t)$ can be of the same sign for all large t. Thus moreover, $\lim_{t\to+\infty} x^{(i)}(t)=0$, $i=1, 2, \cdots, n-1$. Let us now suppose that $\lim_{t\to+\infty} x(t)=\alpha>0$. Then by use of the continuity of the function g, we obtain

$$(1)\qquad g(\alpha, 0, 0, \cdots, 0)-\varepsilon<g(x(t), x'(t), \cdots, x^{(n-1)}(t)) <g(\alpha, 0, 0, \cdots, 0)+\varepsilon$$

for some fixed $\varepsilon<g(\alpha, 0, 0, \cdots, 0)$, and every $t\geq T_1\geq T$. Consequently, we must have (Švec [8], p. 11)

$$(2)\qquad \begin{aligned} x(t)&=\alpha+\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}p(s)g(x(s), x'(s), \cdots, x^{(n-1)}(s))ds \\ &\geq\alpha+[g(\alpha, 0, 0, \cdots, 0)-\varepsilon]\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}p(s)ds=+\infty, \end{aligned}$$

a contradiction. Q.E.D.

2. Let the differential equation $(*)$ be such that $p(t)$ is positive on I, and the function g satisfies the following Condition (G):

$x_1 g(x_1, x_2, \cdots, x_n)>0$ for every $(x_1, x_2, \cdots, x_n)\in \boldsymbol{R}^n$ with $x_1\neq 0$, and for every $(x_1, x_2, \cdots, x_n)\in \boldsymbol{R}^n$, and every $\lambda\geq K$ (=fixed positive constant), $g(-x_1, -x_2, \cdots, -x_n)=-g(x_1, x_2, \cdots, x_n)$, and $g(\lambda, \lambda x_2, \lambda x_3, \cdots, \lambda x_n)=\lambda^\gamma g(1, x_2, \cdots, x_n)$, where $\gamma=q/r$, q, r odd positive integers relatively prime.

Then, if for a solution $x(t)$ of $(*)$ we have $x(t)\geq K$ for $t\in[T, +\infty)$, $x(t)$ must satisfy the equation

$$(\mathrm{E})\qquad z^{(n)}+p(t)g(1, x'(t)/x(t), \cdots, x^{(n-1)}(t)/x(t))z^\gamma=0,$$

$t\in[T, +\infty)$.

Now we are ready for the following

Theorem 2. *Consider $(*)$ with n even, and moreover,*

(i) $p: I\to \boldsymbol{R}_+$;

(ii) $g: \boldsymbol{R}^n\to\boldsymbol{R}$, *and such that Condition* (G) *is satisfied*;

then under any one of the following conditions, all solutions of $(*)$ *are oscillatory*:

a) $\gamma<1$, $\displaystyle\int_{t_0}^\infty t^{\gamma(n-1)}p(t)dt=+\infty$;

b) $\gamma=1$, $\displaystyle\int_{t_0}^\infty t^{n-1-\varepsilon}p(t)dt=+\infty$, *for some ε with* $0<\varepsilon<1$;

c) $\gamma>1$, $\displaystyle\int_{t_0}^\infty t^{n-1}p(t)dt=+\infty$.

Proof. Suppose that $x(t)$, $t\in[t_1, +\infty)$, $t_1\geq t_0$, is a solution of $(*)$ which is non-oscillatory; then by Theorem 1, $x(t)$ must be unbounded on $[t_1, +\infty)$. Without any loss of generality, we suppose that $x(t)>0$ on $[t_1, +\infty)$, and moreover, $\lim_{t\to+\infty} x(t)=+\infty$ (cf. [7], Corollary). Thus, there exists a $t_2 \geq t_1$ such that $x(t)\geq K$ (K as in Condition (G)) for every $t\in[t_2, +\infty)$. It follows that $x(t)$ satisfies the equation (E) for $t\in[t_2, +\infty)$. However, since $\lim_{t\to+\infty} x^{(k)}/x=0$ ([7] Lemma), $k=1, 2, \cdots, n-1$, there is a $t_3\geq t_2$ and an $\varepsilon<g(1, 0, \cdots, 0)$, such that

(3) $$g(1, 0, \cdots, 0)-\varepsilon<g(1, x'(t)/x(t), \cdots, x^{(n-1)}(t)/x(t)) < g(1, 0, \cdots, 0)+\varepsilon$$

for every $t\geq t_3$. Consequently, if

$$Q(t)=p(t)g(1, x'(t)/x(t), \cdots, x^{(n-1)}(t)/x(t))$$

$t\in[t_3, +\infty)$, then for the equation

(E_1) $$z^{(n)}+Q(t)z^{\gamma}=0, \qquad \gamma<1$$

we have:

(4) $$\int_{t_3}^{\infty} t^{\gamma(n-1)}Q(t)dt\geq[g(1, 0, \cdots, 0)-\varepsilon]\int_{t_3}^{\infty} t^{\gamma(n-1)}p(t)dt=+\infty,$$

which implies (cf. [5], Theorem 1) that all solutions of (E_1) are oscillatory, contradicting the fact that $x(t)$ is a solution of (E_1). Thus, in case a), all solutions of (*) are oscillatory. The cases b), c) can be shown similarly by using the result of Ličko and Švec ([5], Theorem 2) for c), and the corresponding result of Mikusiński ([6], p. 35) for the case b).

3. Now it is natural to expect analogous results to hold when n is odd. The following theorem covers this case, and we omit the proof which is very similar to that of Theorem 2, in the presence of the fact that Onose's Lemma ([7], p. 110) also holds for n odd.

Theorem 3. *Let the differential equation (*) with n odd be such that the functions p, g are as in* (i), (ii) *of Theorem* 2 *respectively. Then under any one of the following conditions, every solution of (*) is oscillatory or tending monotonically to zero as $t\to+\infty$:*

a) $\gamma<1$, $\displaystyle\int_{t_0}^{\infty} t^{\gamma(n-1)}p(t)dt=+\infty$;

b) $\gamma=1$, $\displaystyle\int_{t_0}^{\infty} t^{n-1-\varepsilon}p(t)dt=+\infty$, *for some* $\varepsilon>0$;

c) $\gamma>1$, $\displaystyle\int_{t_0}^{\infty} t^{n-1}p(t)dt=+\infty$.

Remark 1. Theorem 2 has been proved in the case $g\equiv x^{\gamma}$ by Ličko and Švec [5] for $\gamma\gtrless 1$, and by Mikusinski [6] for $\gamma=1$. The corresponding cases with n odd are also studied in the same papers. Onose proved Theorem 2 under the assumptions: $\gamma=2s+1$, s nonnegative integer, the condition (G) is satisfied for any λ, and the function p satisfies

$$\int_{t_0}^{\infty} p(t)dt=+\infty.$$

Remark 2. The homogeneity assumption on g can be replaced by inequalities of the form $g(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) \geq \lambda^r g_1(x_1, x_2, \cdots, x_n)$, with appropriate conditions on the function g_1. Thus, we can slightly weaken our assumptions so that we include Howard's Theorem 2 in [1], as a less sharp special case.

Remark 3. It would be very interesting to know under what additional assumptions on the function g, the conditions of Theorems 2, 3 are also necessary for these theorems to hold. For results in this direction see Švec [8], [9] who has used functional-analytic methods in order to obtain monotone solutions of nth-order equations.

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