132. Real-valued Measurable Cardinals and Σ_1^1-Transcendency of Cardinals*)

By Kanji NAMBA
Department of Mathematics, Tokyo University of Education

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In this paper, we shall prove $\sum_1^1$-transcendency of cardinals[1] under the assumption of existence of real-valued measurable cardinal,[2] applying the results and difinitions used in [2] and [3].

Let I be an ideal over a set A. The equivalence relation between two subsets B and C of A is defined by

$$B\sim C\equiv B-C-B\in I\wedge B-C\in I.$$

By $[B]$ we donote the equivalence class including B. And $[A]$ and $[\phi]$ are sometimes abbreviated as $\mathbf{1}$ and $\mathbf{0}$ respectively. The relation $[B]>[C]$ is defined by $[B]>[C]\equiv BC\notin I\wedge C-B\in I$.

An ideal I is called a-complete if

$$[A_\nu]=\mathbf{0} \text{ for all } \nu<a \text{ implies } [\bigcup_{\nu<a} A_\nu]=\mathbf{0}.$$

The character of I is defined to be the smallest ordinal a such that I is not a-complete, and it is denoted by $\mathrm{ch}(I)$.

An ideal I is called a-saturated if

$$[A_\nu]>\mathbf{0},\ [A_\nu\cap A_\mu]=\mathbf{0} \text{ for all } \nu\neq\mu, \text{ and } \nu,\ \mu<b \text{ imply } b<a.$$

The saturation number of I is defined to be the smallest ordinal a such that I is a-saturated, and it is denoted by $\mathrm{sat}(I)$.

Let I be an ideal over $\aleph_\tau$. And let $\mathfrak{A}$ be a set of functions in $\mathrm{On}^{\aleph_\tau}$ (On is the class of all ordinal numbers). A function f is said to be incompressible (cf. [3]) with respect to $\mathfrak{A}$ if the following conditions are satisfied:

(1) $[\{\nu: g(\nu)<f(\nu)\}]=\mathbf{1}$ for every $g\in\mathfrak{A}$,

(2) if $[\{\nu: h(\nu)<f(\nu)\}]>\mathbf{0}$, then, $[\{\nu: h(\nu)\le g(\nu)\}]>\mathbf{0}$ for some $g\in\mathfrak{A}$.

The following lemma is proved easily. (cf. [3]).

Lemma 1. *Let I be an ideal over $\aleph_\tau$ such that* $\mathrm{sat}(I)\le\mathrm{ch}(I)$ ($\aleph_0<\mathrm{ch}(I)$). *And let $\mathfrak{A}$ be a set of functions in* $\mathrm{On}^{\aleph_\tau}$. *Then there is an incompressible function with respect to $\mathfrak{A}$.*

Now we shall define a function $a^*\in On^{\aleph_\tau}$ by the induction on a as one of incompressible functions with respect to $\{b^*: b<a\}$. And $a^*(\nu)$

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1) Cf. [2], [5].

2) Cf. [3], [6].

is abbreviated as $(a)_\nu$.[3]

Lemma 2. *Let $\aleph_\sigma$ be the smallest cardinal number such that $\max(\aleph_1, \mathrm{sat}(I)) \leq \aleph_{\sigma+1}$. Then there is a set of functions $\mathfrak{A}$ with the following properties:*

(1) $\overline{\overline{\mathfrak{A}}} \leq \aleph_\sigma$,

(2) $\mathfrak{A}$ *is arithmetically closed basis* (cf. [1], [2]),

(3) *if $f \in \mathfrak{A}$, then*

$$[\bigcup_{g \in \mathfrak{A}} \{\nu : f((a_1)_\nu, \cdots, (a_n)_\nu) = (g(a_1, \cdots, a_n))_\nu\}] = \mathbf{1}.$$

Proof. Let f be a function. And we consider the set

$$B_{a_1 \cdots a_n} = \{d : [\{\nu : f((a_1)_\nu, \cdots, (a_n)_\nu) = (d)_\nu\}] > \mathbf{0}\}.$$

Then we have

(1) $\overline{\overline{B_{a_1 \cdots a_n}}} \leq \aleph_\sigma$,

(2) $[\bigcup_{d \in B_{a_1 \cdots a_n}} \{\nu : f((a_1)_\nu, \cdots, (a_n)_\nu) = (d)_\nu\}] = 1.$

Now we put $B_{a_1 \cdots a_n} = \{d_\rho : \rho < \aleph_\sigma\}$. And the functions f_ρ is introduced by

$$f_\rho(a_1, \cdots, a_n) = d_\rho.$$

Then we have

$$[\bigcup_{\rho < \aleph_\sigma} \{\nu : f((a_1)_\nu, \cdots, (a_a)_\nu) = (f_\rho(a_1, \cdots, a_a))_\nu\}] = \mathbf{1}.$$

The required set $\mathfrak{A}$ is easily obtained from this.

Lemma 3. *Let I be a proper ideal on $\aleph_\tau$ with the property $\mathrm{sat}(I) < \mathrm{ch}(I) = \aleph_\tau (> \aleph_0)$. And let the following conditions be satisfied:*

(1) $\overline{\overline{B}}, \overline{\overline{\mathfrak{A}}} < \aleph_\tau$, $\aleph_\tau \in B$,

(2) $\mathfrak{A}$ *is arithmetically closed,*

(3) $[\bigcup_{g \in \mathfrak{A}} \{\nu : f((a_1)_\nu, \cdots, (a_n)_\nu) = (g(a_1, \cdots, a_n))_\nu\}] = \mathbf{1}$ *for $f \in \mathfrak{A}$,*

(4) *B is $\mathfrak{A}$-closed,*

(5) *$a \in B$ and $[\{\nu : (d)_\nu = a\}] > \mathbf{0}$ then $d \in B$.*

Then there is an ordinal a_0 such that

(1) $[B \cup \{a_0\}]_{\mathfrak{A}} \cap a_0 \subset B$,

(2) $\sup(B \cap \aleph_\tau) \leq a_0 < \aleph_\tau$, $a_0 \notin B$,

(3) *if $c \in [B \cup \{a_0\}]_{\mathfrak{A}}$ and $[\{\nu : (d)_\nu = c\}] > 0$ then $d \in [B \cup \{a_0\}]_{\mathfrak{A}}$.*

Proof. We shall first define the sets as follows

$$C_a = \bigcup_{[\{\nu : (d)_\nu = a\}] > 0} \{\nu : (d)_\nu = a\},$$

$$A_{fa \cdots a} = \bigcup_{g \in \mathfrak{A}} \{\nu : f((a_1)_\nu, \cdots, (a_n)_\nu) = (g(a_1, \cdots, a_n))_\nu\}.$$

And the set D is defined by

$$D = \{\nu : (\aleph_\tau)_\nu < \aleph_\tau\} \cap \bigcap_{\substack{a<b \\ a, b \in B}} \{\nu : (a)_\nu < (b)_\nu\} \cap \bigcap_{b \in B} C_a \cap \bigcap_{\substack{a_1, \cdots, a_n \in B \\ f \in \mathfrak{A}}} A_{fa_1 \cdots a_n}.$$

By $\mathrm{ch}(I) = \aleph_\tau$, we have $[D] = \mathbf{1}$. Therefore there is a $\nu_0 \in D$. a_0 is

3) Cf. [2].

defined to be $(\aleph)_{\nu_0}$. (2) is clear by this definition. Now we shall prove (1). Assume that

$$c \in [B \cup \{a_0\}]_{\mathfrak{A}} \cap a_0.$$

Since $\mathfrak{A}$ is closed under substitution, there is a function f such that

$$c = f(b_1, \cdots, b_n, a_0) < a_0.$$

By the definition of ν_0, we have $d_1, \cdots, d_n \in B$ such that

$$\nu_0 \in \{\nu : (d_1)_\nu = b_1\}, \cdots, \nu_0 \in \{\nu : (d_n)_\nu = b_n\}.$$

And we also have

$$\nu_0 \in \bigcup_{g \in \mathfrak{A}} \{\nu : f((d_1)_\nu, \cdots, (d_n)_\nu, (\aleph_\tau)_\nu) = (g(d_1, \cdots, d_n, \aleph_\tau))_\nu\}.$$

Therefore for some $g \in \mathfrak{A}$, we have

$$f(b_1, \cdots, b_n, a_0) = f((d_1)_{\nu_0}, \cdots, (d_n)_{\nu_0}, (\aleph_\tau)_{\nu_0}) = (g(d_1, \cdots, d_n, \aleph_\tau))_{\tau_0}.$$

If $\aleph_\tau \leq g(d_1, \cdots, d_n, \aleph_\tau)$, then $a_0 \leq (f(d_1, \cdots, d_n, \aleph_\tau))_{\nu_0}$. Therefore we have

$$g(d_1, \cdots, d_n, \aleph_\tau) < \aleph_\tau.$$

Hence we obtain that $\nu_0 \in \{\nu : (g(d_1, \cdots, d_n, \aleph_\tau))_\nu = g(d_1, \cdots, d_n, \aleph_\tau)\}$. Namely we have $f(b_1, \cdots, b_n, a_0) = g(d_1, \cdots, d_n, \aleph_\nu) \in B$.

Now we prove (3). Assume that

$$c \in [B \cup \{a_0\}]_{\mathfrak{A}} \text{ and } [\{\nu : (d)_\nu = c\}] > 0.$$

Then there are ordinals $d_1, \cdots, d_n \in B$ such that

$$[\{\nu : c = f((d_1)_\nu, \cdots, (d_n)_\nu, (a_0)_\nu) \text{ and } (d)_\nu = c\}] > 0.$$

By $[\bigcup_{g \in \mathfrak{A}} \{\nu : f((d_1)_\nu, \cdots, (d_n)_\nu, (a_0)_\nu) = (g(d_1, \cdots, d_n, a_0))_\nu\}] = 1$, we have for some $g \in \mathfrak{A}$,

$$[\{\nu : (d)_\nu = (g(d_1, \cdots, d_n, a_0))_\nu\}] > 0.$$

Hence we have $d = g(d_1 \cdots, d_n, a_0) \in [B \cup \{a_0\}]_{\mathfrak{A}}$.

By the same method as in [1], [2] we obtain the following theorem.

Theorem. *Let I be a proper ideal on $\aleph_\tau$ such that* $\mathrm{sat}(I) < \mathrm{ch}(I) = \aleph_\tau (> \aleph_0)$. *Then $\sum_1^1$-transcendency of cardinals is true for every cardinals $\aleph_\sigma > \max(\aleph_0, \mathrm{sat}(I))$. Namely we have, for eaery $\sum_1^1$-formula $P(a\, a_1, \cdots, a_n)$ in $(\aleph_\tau)$.*

$$\forall x_1 \cdots \forall x_n (x_1, \cdots, x_n < \aleph_\sigma \wedge \exists x P(x, x_1, \cdots, x_n)$$
$$\rightarrow \exists x (x < \aleph_\sigma \wedge P(x, x_1, \cdots, x_n))).$$

Corollary. *If $\aleph_\tau$ is real-valued measurable cardinal, then $\sum_1^1$-TC holds in $(\aleph_\tau)$.*

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