129. Rings of Dominant Dimension $\geqq 1$

By Toyonori KATO
College of General Education, Tôhoku University, Sendai

(Comm. by Kenjiro SHODA, M. J. A., Sept. 12, 1968)

Tachikawa [9] was the first to introduce and to study dominant dimensions for algebras. In his paper [9], dominant dimensions play a vital role in classifying QF-3 algebras. The author has found it relevant, in the study of rings with torsionless injective hulls, to introduce new dominant dimension defined by the analogue of Tachikawa [9, p. 249] using torsionless minimal injective resolutions.

In Section 1, we introduce dominant dimension for torsionless modules and then give illustrative examples. In Section 2, we are concerned with rings having dominant dimension $\geqq 1$ and show how to obtain such rings. In Theorem 1, (3), we construct a ring Q having dominant dimension ∞ such that Q_Q is a non-injective non-cogenerator in the category of right Q-modules $\mathcal{M}_Q$. The endomorphism rings of generator-cogenerators are discussed in Section 3. Let R and Q be rings. Denote by $\mathcal{M}_R$ (resp. $\mathcal{L}_Q$) the category of right R-modules (resp. the category of right Q-modules having dominant dimension $\geqq 2$). Then our main Theorem 2 states that Q is the endomorphism ring of a generator-cogenerator in $\mathcal{M}_R$ if and only if Q_Q has dominant dimension $\geqq 2$ and there exists an equivalence $\mathcal{M}_R \sim \mathcal{L}_Q$.

In this paper, rings will have a unit element and modules will be unital. If A_R is a module over a ring R, $E(A_R)$ will denote the injective hull of A_R, and $\mathrm{End}(A_R)$ the endomorphism ring of A_R. We adopt the following notational trick, which will facilitate our study: homomorphisms will be written on the side opposite the scalars.

1. Dominant dimension. Let R be a ring, A_R a right R-module, and let

$$0 \to A \to X_1 \to X_2 \to \cdots \to X_n$$

be a minimal (essential) injective resolution of A. We shall say that A has dominant dimension $\geqq n$ if each X_i is torsionless (denoted by domi. dim. $A \geqq n$). In case domi. dim. $A \geqq n$ and domi. dim. $A \ngeqq n+1$, domi. dim. $A = n$. If domi. dim. $A \geqq n$ for each n, domi. dim. $A = \infty$. In case domi. dim. $A \ngeqq 1$, domi. dim. $A = 0$. In case R is left Artinian, torsionless injective right R-modules are always projective by Chase [3, Theorem 3.3] and hence our dominant dimension exactly coincides with that of Tachikawa [9, p. 249]. We consider some illustrative examples of dominant dimensions:

(1) domi. dim. $Z=0$, where Z denotes the ring of integers.

(2) domi. dim. $A_R \geqq 1$ if and only if $E(A_R)$ is torsionless.

(3) R_R is a cogenerator in $\mathcal{M}_R$ if and only if domi. dim. $A_R=\infty$ for each $A_R \in \mathcal{M}_R$.

(4) domi. dim. $R_R \geqq 1$ if and only if domi. dim. $A_R \geqq 1$ for each torsionless $A_R \in \mathcal{M}_R$ (see Kato [5, Proposition 1]).

(5) Let $A=A_1 \in \mathcal{M}_R$, $A_2=E(A_1)/A_1$, $A_3=E(A_2)/A_2$, $\cdots$, $A_n=E(A_{n-1})/A_{n-1}$. In case domi. dim. $R_R \geqq 1$, domi. dim. $A \geqq n$ if and only if each A_i $(i=1, 2, \cdots, n)$ is torsionless.

Proof of (5). If $0 \to A \to X_1 \to X_2 \to \cdots \to X_n$ is an essential injective resolution of A, it can be imbedded in the following commutative exact diagram

$$\begin{array}{ccccccccc} 0 \to & A & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow \cdots \longrightarrow & X_n \\ & \| & & \wr\wr & & \wr\wr & & \wr\wr \\ 0 \to & A & \to & E(A) & \to & E(A_2) & \to \cdots \to & E(A_n) \end{array}$$

making use of the uniqueness of injective hulls. Thus (5) is straightforward in view of Kato [5, Proposition 1].

2. A construction of rings having dominant dimension $\geqq n$ ($n=1, 2, \infty$). Throughout this section, let R be a ring, U_R a generator in $\mathcal{M}_R$, and $Q=\mathrm{End}(U_R)$ the ring of endomorphisms of U_R.

Theorem 1. *Let R, U_R, and Q be as above. Then*

(1) *domi. dim. $Q_Q \geqq 1$ if and only if $E(U_R) \subsetneq \prod U_R$.*

(2) *domi. dim. $Q_Q \geqq 2$ if and only if $E(U_R) \subsetneq \prod U_R$ and $E(U_R)/U_R \subsetneq \prod U_R$.*

(3) *There exists a ring Q of dominant dimension ∞ such that Q_Q is a non-injective non-cogenerator in $\mathcal{M}_Q$.*

Proof. We begin our proof by introducing covariant morphism functors $H: \mathcal{M}_R \to \mathcal{M}_Q$ and $H^*: \mathcal{M}_Q \to \mathcal{M}_R$ as follows:

$$H(A_R)=\mathrm{Hom}(U_R, A_R)_Q \in \mathcal{M}_Q \text{ for each } A_R \in \mathcal{M}_R,$$

$$H^*(A_Q)=\mathrm{Hom}(({}_QU)^*_Q, A_Q)_R \in \mathcal{M}_R \text{ for each } A_Q \in \mathcal{M}_Q$$

where $({}_QU)^*={}_R\mathrm{Hom}({}_QU, {}_QQ)_Q$. Note here that ${}_QU_R$, ${}_R({}_QU)^*_Q$. It is just the point that ${}_QU$ is finitely generated projective and $R=\mathrm{End}({}_QU)$ (see Morita [7, Lemma 3.3]). Therefore, we have

$${}_R(({}_QU)^* \otimes_Q U)_R \approx {}_R\mathrm{Hom}({}_QU, {}_QU)_R \approx {}_RR_R.$$

This isomorphism establishes, for every $A_R \in \mathcal{M}_R$

$$H^*H(A_R)=\mathrm{Hom}(({}_QU)^*_Q, \mathrm{Hom}(U_R, A_R)_Q)_R \approx \mathrm{Hom}(({}_QU)^* \otimes_Q U_R, A_R)_R$$
$$\approx \mathrm{Hom}(R_R, A_R)_R \approx A_R.$$

Thus H^*H is natural equivalent to the identity functor on $\mathcal{M}_R$. Next, H preserves not only injectives by virtue of the projectivity of ${}_QU$ (for the proof, see Cartan and Eilenberg [2, Proposition 1.4, p. 107]) but also essential extensions (see Mitchell [6, Lemma 3.1, p. 88] or C. L. Walker and E. A. Walker [10, Proposition 3.2]). Therefore H

preserves injective hulls. Furthermore, for each $A_R \in \mathcal{M}_R$, $H(A_R)$ is torsionless if and only if $A_R \subsetneq \prod U_R$. To see this, let $A_R \subsetneq \prod U_R$. Then $H(A_R) \subsetneq H(\prod U_R) \approx \prod H(U_R) = \prod Q_Q$. Conversely, let $H(A_R) \subsetneq \prod Q_Q$. Then $A_R \approx H^*H(A_R) \subsetneq H^*(\prod Q_Q) \approx \prod H^*(Q_Q) = \prod H^*H(U_R) \approx \prod U_R$. We are now ready to show (1), (2), and (3).

(1) Since $E(Q_Q) = E(H(U_R)) = H(E(U_R))$, $E(Q_Q)$ is torsionless if and only if $E(U_R) \subsetneq \prod U_R$.

(2) The exact sequence $0 \to Q_Q \to E(Q_Q) \to E(Q_Q)/Q_Q \to 0$ yields an exact commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H^*(Q_Q) & \to & H^*(E(Q_Q)) & \to & H^*(E(Q_Q)/Q_Q) & \to 0 \\ & \wr\wr & & \wr\wr & & & \\ 0 \longrightarrow & U_R & \longrightarrow & E(U_R) & \longrightarrow & E(U_R)/U_R & \longrightarrow 0. \end{array}$$

Thus $E(U_R)/U_R \approx H^*(E(Q_Q)/Q_Q)$. In a similar manner, we have $E(Q_Q)/Q_Q \subsetneq H(E(U_R)/U_R)$. The above justifies that $E(Q_Q) \subsetneq \prod Q_Q$, $E(Q_Q)/Q_Q \subsetneq \prod Q_Q$ if and only if $E(U_R) \subsetneq \prod U_R$, $E(U_R)/U_R \subsetneq \prod U_R$.

(3) Let R_R be a non-injective cogenerator in $\mathcal{M}_R$ (such a ring really exists by an example due to Osofsky [8, Example 2]). Let U_R be infinitely generated free, and let $Q = \mathrm{End}(U_R)$. Then Q_Q has dominant dimension ∞. In fact, let

$$0 \to U_R \to X_1 \to X_2 \to \cdots \to X_n \to \cdots$$

be an essential injective resolution of U_R. Then

$$0 \to H(U_R) \to H(X_1) \to H(X_2) \to \cdots \to H(X_n) \to \cdots$$

is also an essential injective resolution of $H(U_R) = Q_Q$, for H is an exact functor and preserves injective hulls. Now observe that U_R is a cogenerator in $\mathcal{M}_R$ and that $X_i \subsetneq \prod U_R$ for each i. Thus each $H(X_i)$ is torsionless and hence Q_Q has dominant dimension ∞. But Q_Q is non-injective since U_R is non-injective. Moreover Q_Q is a non-cogenerator in $\mathcal{M}_Q$. In fact, ${}_QU$ is a non-generator in ${}_Q\mathcal{M}$ since U_R is infinitely generated (see Morita [7, Lemma 3.3]). Therefore the trace ideal t of ${}_QU$ is proper, but the left annihilator of t is zero by C. L. Walker and E. A. Walker [10, Theorem 1.7]. Thus Q_Q is no more a cogenerator in $\mathcal{M}_Q$.

Remarks. (1) Let the functors H and H^* be as above. Then H^* is the left adjoint to H (see Kan [4]), for making use of the finitely generated projectivity of ${}_QU$ we have

$$\begin{aligned} \mathrm{Hom}(H^*(A_Q), B_R) &= \mathrm{Hom}(\mathrm{Hom}(({}_QU)^*_Q, A_Q)_R, B_R) \\ &= \mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}({}_QU, {}_QQ)_Q, A_Q)_R, B_R) \\ &\approx \mathrm{Hom}(\mathrm{Hom}(Q_Q, A_Q) \otimes_Q U_R, B_R) \\ &\approx \mathrm{Hom}(A_Q \otimes_Q U_R, B_R) \\ &\approx \mathrm{Hom}(A_Q, \mathrm{Hom}(U_R, B_R)_Q) \\ &= \mathrm{Hom}(A_Q, H(B_R)). \end{aligned}$$

(2) In case U_R is a torsionless generator in $\mathcal{M}_R$, domi. dim. $Q_Q \geqq 1$ if and only if domi. dim. $R_R \geqq 1$.

3. A characterization of the endomorphism ring of a generator-cogenerator. Let Q be a ring. We denote by $\mathcal{L}_Q$ the category of right Q-modules having dominant dimension $\geqq 2$.

Theorem 2. *A ring Q is the endomorphism ring of a generator-cogenerator in $\mathcal{M}_R$ if and only if Q_Q has dominant dimension $\geqq 2$ and there exists an equivalence $\mathcal{M}_R \sim \mathcal{L}_Q$.*

Proof. *Sufficiency.* There exist covariant functors $S: \mathcal{M}_R \to \mathcal{L}_Q$ and $T: \mathcal{L}_Q \to \mathcal{M}_R$ such that TS (resp. ST) is natural equivalent to the identity functor on $\mathcal{M}_R$ (resp. $\mathcal{L}_Q$) by virtue of an equivalence $\mathcal{M}_R \sim \mathcal{L}_Q$ (see Mitchell [6, Proposition 10.1, p. 61]). Put $U_R = T(Q_Q)$ making use of $Q_Q \in \mathcal{L}_Q$. Then U_R is a generator in $\mathcal{M}_R$ since Q_Q is a generator in $\mathcal{L}_Q$ (see Bass [1, p. 3]). Since each A_Q in $\mathcal{L}_Q$ is torsionless, Q_Q is a cogenerator in $\mathcal{L}_Q$ and hence $U_R = T(Q_Q)$ is also a cogenerator in $\mathcal{M}_R$ (see Bass [1, p. 3]). Thus U_R is a generator-cogenerator in $\mathcal{M}_R$. Moreover

$$\mathrm{End}(U_R) = \mathrm{End}(T(Q_Q)) \approx \mathrm{End}(Q_Q) \approx Q.$$

Necessity. Let $Q = \mathrm{End}(U_R)$ for a generator-cogenerator U_R in $\mathcal{M}_R$. Then Q_Q has dominant dimension $\geqq 2$ by Theorem 1, (2). Now let the functors H and H^* be as in Proof of Theorem 1. We shall show that H gives the desired equivalence $\mathcal{M}_R \sim \mathcal{L}_Q$. The first part of the proof is to show that H has the codomain $\mathcal{L}_Q$, or equivalently, $H(A_R) \in \mathcal{L}_Q$ for each $A_R \in \mathcal{M}_R$. The next part is to show that $HH^*(A_Q) \approx A_Q$ for each $A_Q \in \mathcal{L}_Q$. These facts together with one established in Proof of Theorem 1 will conclude the proof. Now let $A_R \in \mathcal{M}_R$. Then $A_R \subsetneq \prod U_R$ since U_R is a cogenerator in $\mathcal{M}_R$. Thus $H(A_R)$ is torsionless as was seen in the proof of the preceding theorem. Moreover the same argument as in Theorem 1 yields $E(H(A_R))/H(A_R) \subsetneq H(E(A_R)/A_R)$ and hence $E(H(A_R))/H(A_R)$ is also torsionless. But this implies domi. dim. $H(A_R) \geqq 2$ since Q_Q has dominant dimension $\geqq 2$ by the above. Thus $H(A_R) \in \mathcal{L}_Q$ for each $A_R \in \mathcal{M}_R$. To show that $HH^*(A_Q) \approx A_Q$ for each $A_Q \in \mathcal{L}_Q$, we now define a natural transformation η,

$$\eta_A : A_Q \to HH^*(A_Q) = \mathrm{Hom}(U_R, \mathrm{Hom}(({}_QU)^*_Q, A_Q)_R)_Q$$

by

$$(\eta(a)u)f = a(uf) \quad \text{for } a \in A_Q,\ u \in U_R, \text{ and } f \in ({}_QU)^*.$$

In view of Proof of Theorem 1, we have

$$HH^*(\prod Q_Q) \approx \prod HH^*(Q_Q) \approx \prod HH^*(H(U_R)) \approx \prod H(U_R) = \prod Q_Q$$

and this isomorphism is nothing else but the η. In case A_Q is torsionless, η_A is a monomorphism by the commutativity of the following diagram with an exact row

$$\begin{tikzcd} 0 \arrow[r] & A_Q \arrow[r] \arrow[d, "\eta_A"'] & \prod Q_Q \arrow[d, phantom, "\wr"] \\ & HH^*(A_Q) \arrow[r] & HH^*(\prod Q_Q). \end{tikzcd}$$

Next, let $0 \to A_Q \to \prod Q_Q \to B_Q \to 0$ be an exact sequence with A_Q injective. Notice here that B_Q is torsionless since the above splits. Let us examine the following commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & A_Q \arrow[r] \arrow[d, "\eta_A"'] & \prod Q_Q \arrow[r] \arrow[d, phantom, "\wr"] & B_Q \arrow[r] \arrow[d, "\eta_B"] & 0 \\ 0 \arrow[r] & HH^*(A_Q) \arrow[r] & HH^*(\prod Q_Q) \arrow[r] & HH^*(B_Q) \arrow[r] & 0. \end{tikzcd}$$

Thus η_A is an isomorphism since η_B is a monomorphism (using the Five Lemma). This shows that η_A is an isomorphism in case A_Q is torsionless and injective. Finally, let $A_Q \in \mathcal{L}_Q$. Then we have an exact sequence $0 \to A_Q \to E(A_Q) \to E(A_Q)/A_Q \to 0$ with $E(A_Q)$ and $E(A_Q)/A_Q$ torsionless. This sequence yields the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A_Q \arrow[r] \arrow[d, "\eta_A"'] & E(A_Q) \arrow[r] \arrow[d, phantom, "\wr"] & E(A_Q)/A_Q \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & HH^*(A_Q) \arrow[r] & HH^*(E(A_Q)) \arrow[r] & HH^*(E(A_Q)/A_Q), & \end{tikzcd}$$

concluding that η_A is an isomorphism. Since we have already seen that $H^*H(A_R) \approx A_R$ for each $A_R \in \mathcal{M}_R$ in Theorem 1, we have thus established the equivalence $H: \mathcal{M}_R \sim \mathcal{L}_Q$.

Remarks. (1) For any ring R $\mathcal{M}_R$ has a generator-cogenerator. This observation guarantees the full generality of the above theorem.

(2) Theorem 2 forms an excellent contrast with the following theorem established by Morita [7] (see also Bass [1]):

Morita Theorem. *A ring Q is the endomorphism ring of a progenerator in $\mathcal{M}_R$ if and only if there exists an equivalence $\mathcal{M}_R \sim \mathcal{M}_Q$.*

(3) Monomorphisms, kernels, injective hulls in $\mathcal{L}_Q$ are also the corresponding ones in $\mathcal{M}_Q$. Nevertheless epimorphisms, cokernels, projectives, finitely generated objects in $\mathcal{L}_Q$ do not necessarily coincide with the corresponding ones in $\mathcal{M}_Q$.

Added in proof. At the spring meeting of Math. Soc. Japan in 1968, by Professors K. Morita and H. Tachikawa it was announced that the results [11] related to our Theorem 2 hold for semi-primary QF-3 rings.

After submitting this paper, the author received from Prof. H. Tachikawa a copy of the manuscript [12], in which he had defined the same dominant dimension as ours. He has shown by an example that the two conditions in Theorem 2 are independent for non-semi-primary Q.

Recently Prof. K. Morita has kindly communicated to the author some results connected with Theorem 1, (1), (2). He has studied in the situation that R is right Artinian and U_R is a finitely generated generator in $\mathcal{M}_R$ and then has given an equivalent criterion for the ring $Q=\mathrm{End}(U_R)$ to be of dominant dimension $\geqq n$, introducing new *self-injective dimension* of U_R.

References

[1] H. Bass: The Morita theorems. Lecture Notes, University of Oregon (1962).
[2] H. Cartan and S. Eilenberg: Homological Algebra. Princeton University Press (1956).
[3] S. U. Chase: Direct product of modules. Trans. Amer. Math. Soc., **97**, 457–473 (1960).
[4] D. M. Kan: Adjoint functors. Trans. Amer. Math. Soc., **87**, 294–323 (1958).
[5] T. Kato: Torsionless modules. Tôhoku Math. Journ., **20**, 233–242 (1968).
[6] B. Mitchell: Theory of Categories. Acad. Press (1965).
[7] K. Morita: Duality for modules and its applications to the theory of rings with minimum condition. Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, **6**, No. 150, 83–142 (1958).
[8] B. L. Osofsky: A generalization of quasi-Frobenius rings. Journ. Algebra, **4**, 373–387 (1966).
[9] H. Tachikawa: On dominant dimensions of QF-3 algebras. Trans. Amer. Math. Soc., **112**, 249–266 (1964).
[10] C. L. Walker and E. A. Walker: Quotient Categories of Modules. Proceedings of the La Jolla Conference on Categorical Algebra. Springer Berlin, pp. 404–420 (1966).
[11] K. Morita and H. Tachikawa: On semi-primary QF-3 rings (to appear).
[12] H. Tachikawa: A note on QF-3 rings (to appear).