128. A Milnor Conjecture on Spin Structures

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Let ξ denote a principal $SO(n)$-bundle over a CW-complex B and let $E(\xi)$ denote the total space of ξ. A spin structure on ξ is a pair (η, f) which satisfies

(1) A principal bundle η over B with the spinor group Spin(n) as structural group; and

(2) A map $f: E(\eta)\to E(\xi)$ such that the following diagram is commutative.

$$\begin{array}{ccc} E(\eta)\times \mathrm{Spin}(n) & \rightarrow & E(\eta) \\ \downarrow{\scriptstyle f\times\lambda} & & \downarrow{\scriptstyle f} \\ E(\xi)\times SO(n) & \longrightarrow & E(\xi) \end{array} \quad \begin{array}{c}\searrow \\ \nearrow\end{array} B.$$

Here λ denotes the standard homomorphism from Spin(n) to $SO(n)$ and horizontal lines denote the right translation. A second spin structure (η', f') on ξ is identified with (η, f) if there exists an isomorphism g from η' to η so that $f\circ g=f'$. Then J. Milnor stated the following conjecture [1, pp. 198–203]:

If (η, f) and (η', f') are two spin structures on the same $SO(n)$-bundle, with $n>\dim B$, then η is necessarily isomorphic to η'.

In this note we shall present the affirmative answer when B is compact connected. By Milnor we have the following

Lemma [1, p. 199]: If ξ admits a spin structure then the number of distinct spin structures on ξ is equal to the number of elements in $H^1(B; Z_2)$.

Now the following lemma is clear.

Lemma 1. *If ξ admits two spin structures (η, f) and (η', f') such that η is isomorphic to η' then there exists a spin structure (η, f'') on ξ which is isomorphic to (η', f').*

Let p_ξ denote the projection map of the bundle ξ. If two spin structures (η, f_1), (η, f_2) are given, from $p_\eta=p_\xi f_1=p_\xi f_2$, we have a map $g: E(\eta)\to SO(n)$ defined by $f_1(x)=f_2(x)\cdot g(x)$ for $x\in E(\eta)$. Here $\cdot$ denotes the right translation. Clearly g satisfies $g(x\cdot h)=\lambda(h)^{-1}\times g(x)\times\lambda(h)$ for $h\in\mathrm{Spin}(n)$ where $\times$ denotes the group multiplication. Conversely g is a map as above and let (η, f) be a spin structure on ξ. Then $(\eta, f\cdot g)$[1] is also a spin structure on ξ. And moreover let g' be another map such as g. Then $(\eta, f\cdot g)$ is isomorphic to $(\eta, f\cdot g')$ if

1) Of course the map $f\cdot g$ is defined by $(f\cdot g)(x)=f(x)\cdot g(z)$.

and only if there exists a map $\varphi : E(\eta)\to\mathrm{Spin}(n)$ which satisfies $\varphi(x\cdot h)=h^{-1}\times\varphi(x)\times h$ and $g(x)=g'(x)\times\lambda(\varphi(x))$. Now we define two groups $\langle E(\eta), SO(n)\rangle$ and $\langle E(\eta), \mathrm{Spin}(n)\rangle$ as follows:

$$\langle E(\eta), SO(n)\rangle=\{g : E(\eta)\to SO(n),\ g(x\cdot h)=\lambda(h)^{-1}\times g(x)\times\lambda(h)\}$$

$$\langle E(\eta), \mathrm{Spin}(n)\rangle=\{\varphi : E(\eta)\to \mathrm{Spin}(n),\ \varphi(x\cdot h)=h^{-1}\times\varphi(x)\times h\}.$$

Obviously λ induces a homomorphism $\lambda_* : \langle E(\eta), \mathrm{Spin}(n)\rangle\to\langle E(\eta), SO(n)\rangle$ and if B is connected λ_* is injective. Let $\langle\eta\rangle$ denote the set of spin structures on ξ having η as the bundle of structures. By the above argument we have

Lemma 2. *The number of $\langle\eta\rangle$ is equal to the number of cosets of $\langle E(\eta), SO(n)\rangle$ by $\lambda_*\langle E(\eta), Spin(n)$.*

Let (η, f_0) be a spin structure on ξ and define the group

$$\langle E(\xi), SO(n)\rangle=\{\psi ; E(\xi)\to SO(n),\ \psi(x\cdot g)=g^{-1}\times\psi(x)\times g\}.$$

It is obvious that f_0 induces the homomorphism $f_{0*} ; \langle E(\xi), SO(n)\rangle\to\langle E(\eta), SO(n)\rangle$ defined by $f_{0*}(\psi)=\psi\circ f_0$. Since the kernel of λ is contained in the center of $\mathrm{Spin}(n)$ we have

Lemma 3. *When B is compact f_{0*} is the isomorphism.*

Now consider the inverse image of $\lambda_*\langle E(\eta), SO(n)\rangle$ by f_{0*}. Let $\langle\!\langle E(\xi), SO(n)\rangle\!\rangle$ denote the subgroup of $\langle E(\xi), SO(n)\rangle$ consisting on elements which have a lifting: $E(\xi)\to\mathrm{Spin}(n)$. Then analogously to Lemma 3 we have

Lemma 4. $f_{0*}\langle\!\langle E(\xi), SO(n)\rangle\!\rangle=\lambda_*\langle\!\langle E(\eta), \mathrm{Spin}(n)\rangle\!\rangle$.

Combining Milnor's lemma with the above lemmas we have

Lemma 5. *When B is compact and connected the number of elements of $\mathcal{H}^1(B, \mathbf{Z}_2)$ is equal to the product of the number of cosets of $\langle E(\xi), SO(n)\rangle$ by $\langle\!\langle E(\xi), SO(n)\rangle\!\rangle$ with the number of bundles which give a spin structure on ξ.*

Let B_G denote the classifying space for a topological group G and let x_ξ denote the characteristic map: $B\to B_G$ for a G-bundle ξ. The homomorphism $\lambda : \mathrm{Spin}(n)\to SO(n)$ usually induces the correspondence $B_\lambda : \pi(B, B_{\mathrm{Spin}(n)})\to\pi(B, B_{SO(n)})$. Then it is clear that the number of the inverse image of x_ξ by B_λ is equal to the number of bundles which give a spin structure on ξ. If n is larger than $\dim B$, then $\pi(B, B_{SO(n)})$, $\pi(B, B_{\mathrm{Spin}(n)})$ are equal to $\pi(B, B_{SO(\infty)})$, $\pi(B, B_{\mathrm{Spin}(\infty)})$ respectively. Hence we give a group structure to $\pi(B, B_{\mathrm{Spin}(n)})$ and $\pi(B, B_{SO(n)})$ so that B_λ is a homomorphism. These considerations show that the number of bundles which give a spin structure on ξ is independent on ξ, therefore the number of cosets of $\langle E(\xi), SO(n)\rangle$ by $\langle\!\langle E(\xi), SO(n)\rangle\!\rangle$ is also free from ξ. That is to say the case is only necessary for our purpose that ξ is trivial. Now we suppose that ξ is trivial. Let $\{B, SO(n)\}$ denote the group consisting on all maps: $B\to SO(n)$ and let ρ denote the standard cross-section: $B\to E(\xi)$. It is easily shown that the homomorphism $\rho_*: \langle E(\xi), SO(n)\rangle \to \{B, SO(n)\}$ is bijective where ρ_* is defined by $\rho_*(\phi)=\phi\circ\rho$. Clearly $\rho_*\langle\!\langle E(\xi), SO(n)\rangle\!\rangle$ is contained in $\lambda_*\{B, \mathrm{Spin}(n)\}$.

Conversely, for a map $\lambda\psi$, $\psi: B\to\mathrm{Spin}(n)$, define a map $\phi: E(\xi)\to SO(n)$ by $\phi(b, g)=g^{-1}\times\lambda(\psi(b))\times g$. Then ϕ is an element of $\langle E(\xi), SO(n)\rangle$ such that $\rho_*(\phi)=\lambda\psi$. Let $\tilde{\psi}$ be a map: $E(\xi)\to\mathrm{Spin}(n)$ defined by $\tilde{\psi}(b, g)=h^{-1}\times\psi(b)\times h$ for $\lambda(h)=g$. Since the kernel of λ is contained in the center of Spin(n) $\tilde{\psi}$ is well defined and continuous. By $\lambda\tilde{\psi}=\phi$ we can know that ϕ is an element of $\langle\!\langle E(\xi), SO(n)\rangle\!\rangle$, i.e., we have

Lemma 6. *ρ_* is bijective and maps the subgroup $\langle\!\langle E(\xi), SO(n)\rangle\!\rangle$ onto the subgroup $\lambda_*\{B, Spin(n)\}$.*

Let X_λ[2] denote the cohomology class of $\mathscr{H}^1(SO(n); \boldsymbol{Z}_2)$ which represents the $\boldsymbol{Z}_2$-bundle Spin$(n)\to SO(n)$. Consider a homomorphism $\Phi: \{B, SO(n)\}\to\mathscr{H}^1(B, \boldsymbol{Z}_2)$ defined by $\Phi(\phi)=\phi^*(X_\lambda)$. Now we suppose that $\Phi(\phi)=0$. It is known that if we identify $\mathscr{H}^1(SO(n), \boldsymbol{Z}_2)$ with $\mathrm{Hom}(\pi_1(SO(n)), \pi_1(SO(n)))$ X_λ is correspond to the identity. Since B is connected, we can also identify $\mathscr{H}^1(B, \boldsymbol{Z}_2)$ with $\mathrm{Hom}(H_1(B), \pi_1(SO(n)))$. Then $\phi^*(X_\lambda)$ is interpreted as the composite homomorphism:

$$\mathscr{H}_1(B)\xrightarrow[\phi_*]{}\mathscr{H}_1(SO(n))\xleftarrow[iso]{}\pi_1(SO(n))\xrightarrow[id]{}\pi_1(SO(n)).$$

Hence $\Phi(\phi)=0$ implies that the homomorphism $\phi_*: \pi_1(B)\to\pi_1(SO(n))$ is trivial, i.e., ϕ can be lifted. Hence we have

Lemma 7. *Φ induces the injection:*

$$\{B, SO(n)\}/\lambda_*\{B, \mathrm{Spin}(n)\}\to\mathscr{H}^1(B; \boldsymbol{Z}_2).$$

If $n>\dim B$ we can take the real projective space PR^{n-1} as the classifying space for $\boldsymbol{Z}_2$-bundles over B. On the other hand [2, p. 97] there exists an imbedding $P_n: PR^{n-1}\to SO(n)$ such that $P_n^*: \mathscr{H}^1(SO(n); \boldsymbol{Z}_2)\to\mathscr{H}^1(PR^{n-1}; \boldsymbol{Z}_2)$ is bijective. Thus we have

Lemma 8. *$\Phi: \{B, SO(n)\}/\lambda_*\{B, Spin(n)\}\to\mathscr{H}^1(B; \boldsymbol{Z}_2)$ is bijective.*

From lemmas we obtain our main theorem.

Theorem. *Let B be a compact connected CW-complex. If a principale $SO(n)$-bundle over B admits two spin structures (η, f) and (η', f'), with $n>\dim B$, η is necessary isomorphic to η'.*

References

[1] J. Milnor: Spin structures on manifolds. L'Enseignement Math., **9** (1963).
[2] I. Yokota: J. of Inst. of Poly., Osaka City Univ., **8** (1957).

2) Non-zero element of $\mathscr{H}^1(SO(n); \boldsymbol{Z}_2)\cong\boldsymbol{Z}_2$.