174. An Algebraic Formulation of K-N Propositional Calculus. IV

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In his paper [1], K. Iséki has defined the NK-algebra. For the details of the NK-algebra, see [1]. The conditions of the NK-algebra are as follows:

a) $\sim(p*p)*p=0$,

b) $\sim p*(q*p)=0$,

c) $\sim\sim(\sim\sim(p*r)*\sim(r*q))*\sim(\sim q*p)=0$,

d) Let α, β be expressions in this system, then $\sim\sim\beta*\sim\alpha=0$ and $\alpha=0$ imply $\beta=0$.

In my paper [2], I showed that the NK-algebra is characterized by the following conditions:

a) $\sim(p*p)*p=0$,

b′) $\sim q*(q*p)=0$,

c) $\sim\sim(\sim\sim(p*r)*\sim(r*q))*\sim(\sim q*p)=0$,

d) Let α, β be expressions in this system, then $\sim\sim\beta*\sim\alpha=0$ and $\alpha=0$ imply $\beta=0$.

1) $\sim(p*p)*p=0$.

2) $p*(q*\sim p)=0$.

3) $\sim\sim(\sim\sim(p*r)*\sim(r*q))*\sim(\sim q*p)=0$.

4) $\sim\sim\beta*\sim\alpha=0$ and $\alpha=0$ imply $\beta=0$, where α, β, are expressions in this system. We shall show that 1)–4) imply b′).

In 3), put $p=\beta$, $q=\alpha$, $r=\gamma$, then by 4) we have

A) $\sim\alpha*\beta=0$ implies $\sim\sim(\beta*\gamma)*\sim(\gamma*\alpha)=0$.

Then we have

B) $\sim\alpha*\beta=0$ and $\gamma*\alpha=0$ imply $\beta*\gamma=0$.

C) $\sim\alpha*\beta=0$ and $\sim\gamma*\alpha=0$ imply $\beta*\sim\gamma=0$.

In B), put $\alpha=\sim p*\sim p$, $\beta=\sim p$, $\gamma=p$, then by 1) and by 2) we have

5) $\sim p*p=0$.

In 3), put $q=p$, then $\sim\sim(\sim\sim(p*r)*\sim(r*p))*\sim(\sim p*p)=0$. By 5) we have

6) $\sim\sim(p*r)*\sim(r*p)=0$.

In 6), put $p=\alpha$, $r=\beta$, then $\sim\sim(\alpha*\beta)*\sim(\beta*\alpha)=0$. Hence by 4) we have

D) $\beta*\alpha=0$ implies $\alpha*\beta=0$.

In 6), put $r=\sim q$, then by 5) we have

7) $q*\sim q=0$.

In 3) $p=\sim\beta$, $q=\sim\alpha$, $r=\alpha$, then $\sim\sim(\sim\sim(\sim\beta*\alpha)*\sim(\alpha*\sim\alpha))*\sim(\sim\sim\alpha*\sim\beta)=0$. By 7) we have

E) $\sim\sim\alpha*\sim\beta=0$ implies $\sim\beta*\alpha=0$.

In 3, put $p=\alpha$, $q=\beta$, $r=\gamma$, then we have $\sim\sim(\sim\sim(\alpha*\gamma)*\sim(\gamma*\beta))*\sim(\sim\beta*\alpha)=0$. By E) and 4) we have

F) $\sim\beta*\alpha=0$ implies $\sim(\gamma*\beta)*(\alpha*\gamma)=0$.

In F), put $\alpha=q$, $\beta=\sim\sim q$, $\gamma=p$, then by 7) and E) we have

8) $\sim(p*\sim\sim q)*(q*p)=0$.

Suppose $\sim\alpha*\beta=0$ and $\sim\gamma*\alpha=0$, then by C) we have $\beta*\sim\gamma=0$. Further by D), $\beta*\sim\gamma=0$ implies $\sim\gamma*\beta=0$. Hence we have

G) $\sim\alpha*\beta=0$, $\sim\gamma*\alpha=0$ imply $\sim\gamma*\beta=0$.

In 2), put $p=\sim q$, $q=p$, then $\sim q*(p*\sim\sim q)=0$. Hence by 8), 2) and G) we have

9) $\sim q*(q*p)=0$.

This thesis is b′). Therefore the proof is complete.

References

[1] K. Iséki: An algebraic formulation of *K-N* propositional calculus. Proc. Japan Acad., **42**, 1164–1167 (1966).

[2] S. Tanaka: An algebraic formulation of *K-N* propositional calculus. II. Proc. Japan Acad., **43**, 129–131 (1967).