# 209. *A Simple Characterization of Boolean Rings*

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(Comm. by Kinjirô KUNUGI, M. J. A., Nov. 12, 1968)

G. R. Blakley, S. Ôhashi, and the present author give some new axioms for commutative rings (see [1]-[4]). In this Note, we shall give a new axiom system of Boolean rings.

Let $<R, +, \cdot, -, 0, 1>$ be an algebraic system, where $R$ is a non-empty, 0 and 1 are elements of $R$, $+$, and . are binary operations on $R$, and $-$ is a unary operation on $R$.

Then we have the following

**Theorem.** *$<R, +, \cdot, -, 0, 1>$ is a Boolean ring, if it satisfies the following conditions*:

1) $r=r+0,$

2) $r1=1r=r,$

3) $((-r)+r)a=0,$

4) $((ay+bx)+cr)r=b(rx)+(a(yr)+cr).$

In our discussion, we do not use the multiplication symbol dot. Therefore $ab$ means $a\cdot b$.

The proof of Theorem follows from the following several steps.

5) $(-r)+r=0.$

$0=((-r)+r)1$ {3}

$=(-r)+r.$ {2}

6) $0a=0.$

$0a=((-r)+r)a=0.$ {5, 3}

7) $a+b=b+a.$

$a+b=((a1+b1)+01)1$ {1, 2, 6}

$=b(11)+(a(11)+01)$ {4}

$=b+a.$ {1, 2, 6}

8) $(ay)r=a(yr).$

$(ay)r=((ay+0x)+0r)r$ {1, 6}

$=0(rx)+(a(yr)+0r)$ {4}

$=a(yr).$ {1, 6, 7}

9) $(a+b)+c=a+(b+c).$

$(a+b)+c=(b+a)+c$ {7}

$=((b1+a1)+c1)1$ {2}

$=a(11)+(b(11)+c1)$ {4}

$=a+(b+c).$ {2}

10) $(a+b)r=ar+br.$

$$(a+b)r=((a1+b1)+0r)r \quad \{1, 2, 6\}$$
$$=b(r1)+(a(1r)+0r) \quad \{4\}$$
$$=br+ar \quad \{1, 2, 6\}$$
$$=ar+br. \quad \{7\}$$

11) $ab=ba$.

$$ab=((0y+1a)+0b)b \quad \{1, 2, 7\}$$
$$=1(ba)+(0(yb)+0b) \quad \{4\}$$
$$=ba. \quad \{1, 2, 6\}$$

12) $a^2=a$, i.e., $aa=a$.

$$aa=((0y+0x)+1a)a \quad \{1, 6, 7\}$$
$$=0(ax)+(0(ya)+1a) \quad \{4\}$$
$$=1a \quad \{1, 6, 7\}$$
$$=a. \quad \{2\}$$

13) For given $a$, $b$ the equation $a+x=b$ is solvable.

Let $x=(-a)+b$, then we have $a+x=a+((-a)+b)=(a+(-a))+b =((-a)+a)+b=0+b=b$. Hence $(-a)+b$ is the solution of the equation.

Therefore we have $c(a+b)=ca+cb$. This means that $R$ is a ring. As is shown in 12), every element of $R$ is the idempotent, therefore $R$ is a Boolean ring.

## References


[1] G. R. Blakley: Four axioms for commutative rings. Notices of Amer. Math. Soc., **15**, p. 730 (1968).

[2] K. Iséki and S. Ôhashi: On definitions of commutative rings. Proc. Japan Acad., **44**, 920–922 (1968).

[3] S. Ôhashi: On axiom systems of commutative rings. Proc. Japan Acad., **44**, 915–919 (1968).