206. Generalized Product and Sum Theorems for Whitehead Torsion

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1. Introduction. Let K and L be finite CW-complexes and let $f: K \to L$ be a cellular map. If f is a homotopy equivalence, the Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi)$ is defined, where $\mathrm{Wh}(\pi)$ is the Whitehead group of the fundamental group π of L (for the definitions, see Milnor [2]).

Whitehead has proved in [4] that K and L are of the same simple homotopy type iff there is a homotopy equivalence $f: K \to L$ such that $\tau(f)=0$.

In 1965, Kwun and Szczarba proved two theorems for Whitehead torsion [1]; one is the Sum Theorem, and the other the Product Theorem. The Sum Theorem is stated as follows.

Theorem I. *Let X and Y be finite cell complexes which are the union of subcomplexes $X=X_1 \cup X_2$, $Y=Y_1 \cup Y_2$, and X_0, Y_0 the intersection $X_0=X_1 \cap X_2$, $Y_0=Y_1 \cap Y_2$. Let $f: X \to Y$ be a cellular map and $f \mid X_i=f_i: X_i \to Y_i$ $(i=0, 1, 2)$. If f_i are homotopy equivalences and X_0 is connected and simply connected, then f is a homotopy equivalence and*

$$\tau(f)=j_{1*}\tau(f_1)+j_{2*}\tau(f_2), \tag{1}$$

where $j_{i*}: \mathrm{Wh}(\pi_1(Y_i)) \to \mathrm{Wh}(\pi_1(Y))$ are induced by the inclusion maps.

In this paper we shall consider the case when X_0 is non-simply connected. Then we obtain the following result which is a generalization of Theorem I.

Theorem I′. *Let X, Y be finite CW-complexes which are the union of subcomplexes $X=X_1 \cup X_2$, $Y=Y_1 \cup Y_2$. Put $X_0=X_1 \cap X_2$, $Y_0=Y_1 \cap Y_2$. Let $f: X \to Y$ be a cellular map and $f_i=f \mid X_i: X_i \to Y_i$ be homotopy equivalences $(i=0, 1, 2)$. If X_0 is connected, then f is a homotopy equivalence and*

$$\tau(f)=j_{1*}\tau(f_1)+j_{2*}\tau(f_2)-j_{0*}\tau(f_0), \tag{2}$$

where $j_i: Y_i \to Y$ are inclusions.

In particular, if X_0 is simply connected, then $\tau(f_0)=0$ and hence we get formula (1) from formula (2).

Next, the Product Theorem in [1] reads as follows.

Theorem II. *If C is an acyclic based A-complex and C' a based B-complex, then $\tau(C \otimes_z C')=\chi(C')i_*\tau(C)$, where $\chi(C')$ is the Euler* *characteristic of* C' *and* $i_*: \bar{K}_1(A) \to \bar{K}_1(A \otimes_z B)$ *is induced by the map* $a \to a \otimes 1$.

J. Milnor defined in [2] the torsion for non-acyclic based complexes. We attempt to calculate the torsion $\tau(C \otimes_z C')$ when C, C' are not necessarily acyclic. We say here that a finite complex C is a *based* A-*complex* if C_q, $H_q(C)$ are free A-modules with prefered bases and $B_q(C) = \partial C_{q+1}$ is also free.

Theorem II′. *If* C *is a based* A-*complex and* C' *a based* B-*complex, then* $C \otimes_z C'$ *is a based* $A \otimes_z B$-*complex and*

$$\tau(C \otimes_z C') = \chi(C) j_* \tau(C') + \chi(C') i_* \tau(C),$$

where the map $j: B \to A \otimes_z B$ *is defined by* $j(b) = 1 \otimes b$, *and* i *as above.*

2. Proof of Theorem I′. In this paper, we use the results of Milnor's paper [2] and his notations. Spaces are connected finite CW-complexes and maps are cellular maps. We shall first prove the following theorem.

Theorem 1. *Let* $f: X \to Y$ *be a homotopy equivalence and let* $X' = X \cup_g D^2$, $Y' = Y \cup_{fg} D^2$, *where* $g: \dot{D}^2 \to X$. *Define* $f': X' \to Y'$ *by* $f' | X = f$, $f' | \operatorname{int} D^2 =$ *identity. Then* f' *is a homotopy equivalence and* $\tau(f') = h_* \tau(f)$, *where* $h: Y \to Y'$ *is the inclusion.*

Proof. It is obvious that f' is a homotopy equivalence. Let $j: Z[\pi_1(X)] \to Z[\pi_1(X')]$ be the ring homomorphism induced by the inclusion map and let $p: \tilde{M}_f \to M_f$, $p': \tilde{M}_{f'} \to M_{f'}$ be the universal coverings of the mapping cylinders of f, f'. Put $p^{-1}(X) = \tilde{X}$, $p'^{-1}(X') = \tilde{X}'$. There is a natural map $p'': \tilde{M}_f \to p'^{-1}(M_f)$ such that $p'p'' = p$. p'' induces a simple isomorphism

$$Z[\pi_1(X')] \otimes_j C(\tilde{M}_f, \tilde{X}) \cong C(p'^{-1}(M_f \cup X'), \tilde{X}').$$

Since each component of $\tilde{M}_{f'} - p'^{-1}(M_f \cup X')$ is simply connected, we have

$$\begin{aligned} \tau(C(\tilde{M}_{f'}, \tilde{X}')) &= \tau(C(\tilde{M}_{f'}, p'^{-1}(M_f \cup X'))) + \tau(C(p'^{-1}(M_f \cup X'), \tilde{X}')) \\ &= \tau(C(p'^{-1}(M_f \cup X'), \tilde{X}')) \\ &= j_* \tau(C(\tilde{M}_f, \tilde{X})). \end{aligned}$$

Therefore $\tau(f') = f'_* \tau(C(\tilde{M}_{f'}, \tilde{X}')) = f'_* j_* \tau(C(\tilde{M}_f, \tilde{X})) = h_* \tau(f)$.

Corollary. *Let* $f: X \to Y$ *be a homotopy equivalence and let* g_i *be maps* $g_i: \dot{D}_i^2 \to X$. *Define*

$$f': X \cup_{g_1} D_1^2 \cup \cdots \cup_{g_r} D_r^2 \to Y \cup_{fg_1} D_1^2 \cup \cdots \cup_{fg_r} D_r^2$$

by $f' | X = f$, $f' | \operatorname{int} D_i^2 =$ *identity. Then* f' *is a homotopy equivalence and* $\tau(f') = h_* \tau(f)$, *where* h *is the inclusion.*

Proof. This is proved by induction on r.

Theorem 2. *If the inclusion map* $X_0 \to X$ *induces a monomorphism* $\pi_1(X_0) \to \pi_1(X)$, *then Theorem* I′ *holds.*

Lemma 1. *Under the same condition as Theorem* 2,

(1) $$\pi_1(X_0) \to \pi_1(X_i) \qquad (i = 1, 2),$$

(2) $$\pi_1(X_i)\to\pi_1(X) \qquad (i=1,2),$$
are monomorphisms.

Proof. (1) is trivial. $\pi_1(X)$ is an amalgamated product of the family $\{\pi_1(X_i), \pi_1(X_0)\to\pi_1(X_i)\}$, hence $\pi_1(X_i)\to\pi_1(X)$ are monomorphisms (A. G. Kurosch, Theory of groups, § 35, Chelsea, 1960).

Let L be a subcomplex of a complex K and $p:\tilde{K}\to K$ be a universal covering of K. Let $\tilde{L}$ be one of the components of $p^{-1}(L)$.

Lemma 2. *If $\pi_1(L)\to\pi_1(K)$ is a monomorphism, then $p'=p\,|\,\tilde{L}:\tilde{L}\to L$ is a universal covering of L.*

Proof. It is sufficient to show that $\tilde{L}$ is simply connected. But this is an immediate consequence of the covering homotopy property.

Proof of Theorem 2. The homotopy equivalence is easily proved.

Let $p:\tilde{M}_f\to M_f$ be the universal covering of the mapping cylinder of f. Since the exact sequence

$$0\to C(p^{-1}(M_{f_0}), p^{-1}(X_0))\xrightarrow{\varphi} C(p^{-1}(M_{f_1}), p^{-1}(X_1))\oplus C(p^{-1}(M_{f_2}), p^{-1}(X_2))$$
$$\xrightarrow{\phi} C(\tilde{M}_f, p^{-1}(X))\to 0,$$

where $\varphi(c)=(c,c)$, $\phi(c_1,c_2)=c_1-c_2$, is compatible for the prefered bases, we have

$$\tau(C(p^{-1}(M_{f_1}), p^{-1}(X_1)))+\tau(C(p^{-1}(M_{f_2}), p^{-1}(X_2)))$$
$$=\tau(C(p^{-1}(M_{f_0}), p^{-1}(X_0)))+\tau(C(\tilde{M}_f, p^{-1}(X))).$$

We have to prove $f_*\tau(C(p^{-1}(M_{f_i}), p^{-1}(X_i)))=j_{i*}\tau(f_i)$ for $i=0,1,2$.

Let $\tilde{M}_{f_i}$ be one of the components of $p^{-1}(M_{f_i})$. Since $\pi_1(M_{f_i})\to\pi_1(M_f)$ is a monomorphism, $p_i=p\,|\,\tilde{M}_{f_i}:\tilde{M}_{f_i}\to M_{f_i}$ is a universal covering. Let $h_i:Z[\pi_1(X_i)]\to Z[\pi_1(X)]$ be a homomorphism induced by the inclusion. Then

$$C(p^{-1}(M_{f_i}), p^{-1}(X_i))\cong Z[\pi_1(X)]\otimes_{h_i}C(\tilde{M}_{f_i}, p_i^{-1}(X_i))$$

is simple isomorphic. Since $f_*h_{i*}=j_{i*}f_{i*}$,

$$f_*\tau(C(p^{-1}(M_{f_i}), p^{-1}(X_i)))=f_*h_{i*}\tau(C(\tilde{M}_{f_i}, p_i^{-1}(X_i))$$
$$=j_{i*}f_{i*}\tau(C(\tilde{M}_{f_i}, p_i^{-1}(X_i)))=j_{i*}(f_i).$$

This completes the proof.

Proof of Theorem I′. Let $g_i:\dot{D}_i^2\to X_0$, $i=1,\cdots,r$ be representations for generators of $\mathrm{Ker}\,(\pi_1(X_0)\to\pi_1(X))$ and let $k_i:X_0\to X_i$ be inclusions. Put X'_i, Y'_i $(i=0,1,2)$ as $X'_i=X_i\cup_{k_ig_1}D_1^2\cup\cdots\cup_{k_ig_r}D_r^2$, $Y'_i=Y_i\cup_{f_ik_ig_1}D_1^2\cup\cdots\cup_{f_ik_ig_r}D_r^2$ and $X'=X'_1\cup X'_2$, $Y'=Y'_1\cup Y'_2$. Define $f'_i:X'_i\to Y'_i$, $f':X'\to Y'$ as similarly defined in the corollary of Theorem 1. Clearly X', Y', X'_i, Y'_i, f', f'_i satisfy the conditions of Theorem 2. Hence

$$\tau(f')=j'_{1*}\tau(f'_1)+j'_{2*}\tau(f'_2)-j'_{0*}\tau(f'_0),$$

where $j'_i:Y'_i\to Y'$ are inclusions. Let $h:Y\to Y'$ be the inclusion. By Corollary to Theorem 1, $\tau(f')=h_*\tau(f)$, $j'_{i*}\tau(f'_i)=h_*j_{i*}\tau(f_i)$ $(i=0,1,2)$. Therefore

$$h_*\tau(f)=h_*(j_{1*}\tau(f_1)+j_{2*}\tau(f_2)-j_{0*}\tau(f_0)).$$

Since $fkg_i\simeq 0$, where $k: X_0\to X$ is the inclusion, $\pi_1(Y)\to\pi_1(Y')$ is an isomorphism and so is $h_*: \mathrm{Wh}(\pi_1(Y))\to \mathrm{Wh}(\pi_1(Y'))$. Hence the Theorem I′ holds.

3. Proof of Theorem II′. If X is a free A-module and Y a free B-module with bases $x=(x^1, \cdots, x^r)$ and $y=(y^1, \cdots, y^s)$ respectively, then $X\otimes_z Y$ is a free $A\otimes_z B$-module with base $x\otimes y=(x^1\otimes y^1, x^1\otimes y^2, \cdots \cdots, x^r\otimes y^s)$, and if $A=B$, direct sum $X\oplus Y$ is a free A-module with base $xy=(x^1, \cdots, y^s)$.

Lemma 3. *Let u, u', u_1u_2 be three bases for free A-module X and v, v', v_1v_2 be those for free B-module Y. Then*

(1) $$[u\otimes v/u\otimes v']=\alpha(X)j_*[v/v'],$$

(2) $$[u\otimes v/u'\otimes v]=\alpha(Y)i_*[u/u'],$$

(3) $$[(u\otimes v_1)(u\otimes v_2)/u\otimes(v_1v_2)]=0,$$

(4) $$[(u_1\otimes v)(u_2\otimes v)/(u_1u_2)\otimes v]=0,$$

where i_*, j_* are the same as in Introduction and $\alpha(G)=$(the minimum of the number of generators of G).

Proof. If $u=(u^1, \cdots, u^r)$, $v=(v^1, \cdots, v^s)$, $v'=(v'^1, \cdots, v'^s)$ and $v^k=\sum_j x_{k,j}v'^j$, $x_{k,j}\in B$, then $u^p\otimes v^q=\sum_j(1\otimes x_{q,j})u^p\otimes v'^j$. Let T be a $s\times s$ matrix such that $(T)_{i,j}=1\otimes x_{i,j}$. Then

$$u\otimes v/u\otimes v'=\begin{pmatrix} T & & & 0\\ & T & & \\ & & \ddots & \\ 0 & & & T\end{pmatrix},$$

hence $[u\otimes v/u\otimes v']=r[T]=\alpha(X)j_*[v/v']$.

(2) is proved similarly and (3), (4) are permutations of bases.

Proof of Theorem II′. Let c_q, h_q be the prefered bases of C_q, $H_q(C)$ and c'_q, h'_q be those of C'. By the Künneth formula, $C\otimes_z C'$ is a based $A\otimes_z B$-complex with prefered bases $(c_0\otimes c'_q)(c_1\otimes c'_{q-1})\cdots(c_q\otimes c'_0)$, $(h_0\otimes h'_q)(h_1\otimes h'_{q-1})\cdots(h_q\otimes h'_0)$. Let C' be the form

$$C'_p\to C'_{p-1}\to\cdots\to C'_q\to 0.$$

We proceed by induction on $p-q$.

If $p-q=0$, then $(C\otimes C')_i=C_{i-q}\otimes C'_q$, $H_i(C\otimes C')=H_{i-q}(C)\otimes H_q(C')$, having the bases $c_{i-q}\otimes c'_q$, $h_{i-q}\otimes h'_q$. Choose a base b_r of $B_r=\partial C_{r+1}$ for each r. We can choose a base $b_{i-q}\otimes c'_q$ of $B_i(C\otimes C')$ for each r. By Lemma 3,

$$\begin{aligned}&[(b_r\otimes c'_q)(h_r\otimes h'_q)(b_{r-1}\otimes c'_q)/c_r\otimes c'_q]\\ &=[(b_r\otimes c'_q)(h_r\otimes c'_q)(b_{r-1}\otimes c'_q)/c_r\otimes c'_q]+[h_r\otimes h'_q/h_r\otimes c'_q]\\ &=\alpha(C'_q)i_*[b_rh_rb_{r-1}/c_r]+\alpha(H_r(C))j_*[h'_q/c'_q].\end{aligned}$$

Therefore

$$\begin{aligned}\tau(C\otimes C')&=\textstyle\sum_r(-1)^{q+r}\{\alpha(C'_q)i_*[b_rh_rb_{r-1}/c_r]+\alpha(H_r(C))j_*[h'_q/c'_q]\}\\ &=(-1)^q\alpha(C'_q)i_*\textstyle\sum_r(-1)^r[b_rh_rb_{r-1}/c_r]\\ &\quad+\{\textstyle\sum_r(-1)^r\alpha(H_r(C))\}j_*(-1)^q[h'_q/c'_q]\\ &=\chi(C)j_*\tau(C')+\chi(C')i_*\tau(C).\end{aligned}$$

When $p-q\geqq 1$, let D, D' be the chain complexes $C'_q\to 0$ and $C'_p\to C'_{p-1}\to\cdots\to C'_{q+1}\to 0$. Then $H_q(D)\cong C'_q$, $H_{q+1}(D')\cong C'_{q+1}/B'_{q+1}$ are free. (B'_{r-1} is free and $0\to Z'_r/B'_r\to C'_r/B'_r\to C'_r/Z'_r\cong B'_{r-1}\to 0$ splits, hence $C'_r/B'_r\cong H'_r\oplus B'_{r-1}$.) Let x, y be their bases. Since the other bases are induced from those of C', we can regard D, D' as the based B-complexes. The exact sequence

$$0\to C\otimes D\to C\otimes C'\to C\otimes D'\to 0$$

is compatible with respect to these prefered bases. Denote the homology sequence induced by the above sequence by $\mathcal{H}$. By Milnor [2, Theorem 3.2] and by the assumption of induction,

$$\begin{aligned}\tau(C\otimes C')&=\tau(C\otimes D)+\tau(C\otimes D')+\tau(\mathcal{H})\\&=\chi(C)j_*\tau(D)+\chi(D)i_*\tau(C)+\chi(C)j_*\tau(D')+\chi(D')i_*\tau(C)+\tau(\mathcal{H})\\&=\chi(C)j_*(\tau(D)+\tau(D'))+\chi(C')i_*\tau(C)+\tau(\mathcal{H}).\end{aligned}$$

A tedious but not difficult calculation shows that

$$\tau(\mathcal{H})=\chi(C)j_*(-1)^q\{[b'_qh'_q/x]-[h'_{q+1}b'_q/y]\}.$$

On the other hand,

$$\begin{aligned}\tau(C')-\tau(D)-\tau(D')&=\textstyle\sum_i(-1)^i[b'_ih'_ib'_{i-1}/c'_i]-(-1)^q[x/c'_q]\\&\quad-\textstyle\sum_{i=q+2}^{p}(-1)^i[b'_ih'_ib'_{i-1}/c'_i]-(-1)^{q+1}[b'_{q+1}y/c'_{q+1}]\\&=(-1)^q\{[b'_qh'_q/c'_q]-[b'_{q+1}h'_{q+1}b'_q/c'_{q+1}]-[x/c'_q]+[b'_{q+1}y/c'_{q+1}]\}\\&=(-1)^q\{[b'_qh'_q/x]-[h'_{q+1}b'_q/y]\}.\end{aligned}$$

Therefore $\tau(C\otimes C')-\chi(C)j_*\tau(C')-\chi(C')i_*\tau(C)=0$.

References

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