205. On Generalized Integrals. III

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In the preceding papers [3], we showed that the special $(E.R.)$ integral is defined as a unique and natural extension of integrals (defined as usual) of step functions, using the method of the ranked space. In fact, to do this, we introduced on the set $\mathcal{E}$ of step functions on $[a, b]$ a set of neighbourhoods, denoted by $V(A, \varepsilon; f)$, and a rank so that $\mathcal{E}$ should become a ranked space. In this ranked space $\mathcal{E}$, we see that if $u: \{V_n(f_n)\}$ is a fundamental sequence of neighbourhoods, the limit $f(x)=\lim_{n\to\infty} f_n(x)$ exists almost everywhere, and the sequence of integrals $\int_a^b f_n(x)dx$ converges to a finite limit. Moreover, if $u: \{V_n(f_n)\}$ and $v: \{V_n(g_n)\}$ are two fundamental sequences belonging to the same maximal collection u^*, then we have

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} g_n(x) \quad \text{a.e.},$$

$$\lim_{n\to\infty}\int_a^b f_n(x)dx=\lim_{n\to\infty}\int_a^b g_n(x)dx.$$

Therefore, each maximal collection u^* in $\mathcal{E}$ determines a function and a value which we can associate to this u^*. $J(u^*)$ denotes the function and $I(u^*)$ denotes the value. If we denote, by $\boldsymbol{U}$, the set of all maximal collections u^*, we have $J(u^*)\neq J(v^*)$ for $u^*\in \boldsymbol{U}$ and $v^*\in \boldsymbol{U}$ such that $u^*\neq v^*$. We denoted, by $\boldsymbol{K}$, the set $\{J(u^*); u^*\in \boldsymbol{U}\}$, and for each $f=J(u^*)$, we defined the integral $I(f)$ of f by taking the value $I(u^*)$. Then, $\boldsymbol{K}$ coincides with the set of $(E.R.)$ integrable functions in the special sense (or A-integrable functions) and we have $I(f)=(E.R.)\int_a^b f(x)dx =(A)\int_a^b f(x)dx$. In this paper, we will show that if we reasonably introduce a set of neighbourhoods and a rank on $\boldsymbol{K}$, then the ranked space $\boldsymbol{K}$ is a completion of the ranked space $\mathcal{E}$ (Theorem 3). Moreover, the special $(E.R.)$ integral is the r-continuous extension of integrals of step functions, and it is a r-continuous linear functional on the complete ranked space $\boldsymbol{K}$ (Theorem 4).

In order to introduce the notion of completion in the ranked spaces,[1] we first recall a few basic concepts in the general ranked spaces. Throughout this paper, we suppose that the ranked spaces

1) For the problem of the completion of the ranked spaces, see [1] and [5].

R satisfy the axiomes (A) and (B) of Hausdorff and have the indicator ω_0 (ω_0 is the first non-finite cardinal). Prof. K. Kunugi in [2] gave the notion of the limit in the ranked space in the following form:

Definition of the convergence of a sequence of points. Given a sequence $\{p_n; n=0, 1, 2, \cdots\}$ of points of R and a point p of R, we say that the sequence $\{p_n\}$ *r-converges* to the point p, or that p is a *r-limit* of $\{p_n\}$, if there is a fundamental sequence $\{V_n(p)\}$ consisting of neighbourhoods of p such that $V_n(p) \ni p_n$ for each n. In this case, we write

$$p \in \{\lim_n p_n\}.$$

$\{\lim_n p_n\}$ is not a set consisting of one point alone in general.

We write $Cl_r(E)$ the set of all r-limit points of a set E [6]. We say that a set E is *r-dense* in a ranked space R if $Cl_r(E)=R$.

Definition of the continuity. Let R, S be two ranked spaces. Consider a one valued function $f(p)$ defined for every $p \in R$ and taking values in a set S. Let $p_0 \in R$. Then, the function $f(p)$ is said to be *r-continuous* at the point p_0 if $p_0 \in \{\lim_n p_n\}$ implies $f(p_0) \in \{\lim_n f(p_n)\}$[2]. The function is said to be r-continuous if it is r-continuous at each point of R. In particular, the continuity of the real valued function $f(p)$ is usually understood, unless the contrary is expressly stated, as follows: $f(p)$ is r-continuous at the point p_0 if $\{\lim_n p_n\} \ni p_0$ implies $\lim_{n\to\infty} f(p_n) = f(p_0)$.

When f is a one-to-one function of R onto S, and both f and f^{-1} is r-continuous, we say that f is a *r-isomorphism* of R into S, and the spaces R and S are said to be *r-equivalent*.

Relative notions. Let A be a subset of R. For every point p of A, the neighbourhood of p in A is the set of points of A defined by the relation $V(p, A) = V(p) \cap A$, where $V(p)$ is a neighbourhood of p in R. We also define the set $\mathfrak{V}_n(A)$ $(n=0, 1, 2, \cdots)$ of neighbourhoods of rank n of points of A as follows: $V(p, A) \in \mathfrak{V}_n(A)$ if and only if $V(p) \in \mathfrak{V}_n$, where $\mathfrak{V}_n$ is a set of neighbourhoods of rank n in R. Then, A is a ranked space. Prof. K. Kunugi called it a *ranked space induced from R* [2].

In particular, let us consider a ranked spaces A induced from R such that: for every $p \in A$ and for every fundamental sequence $\{V_n(p, A)\}$ of neighbourhoods of p, there is a fundamental sequence $\{V_n(p)\}$ of neighbourhoods of p in R for which we have $V_n(p, A) = V_n(p) \cap A$ for each n. Y. Yoshida called this ranked space A a *ranked subspace* of R [6]. He showed that when $\{p_n\}$ is a sequence of points A and p is a point of A, we have $p \in \{\lim_n p_n\}$ in A if and only if $p \in \{\lim_n p_n\}$ in R.

Definition of the complete space.[2] The ranked space R is said to be *complete*, if, for every fundamental sequence $\{V_n(p_n)\,;\, n=0,1,2,\cdots\}$ of neighbourhoods, we have $\bigcap_{n=0}^{\infty} V_n(p_n) \neq \phi$.

Now, we will give a definition of completion of the ranked space.

Definition 2. A ranked space R^* is called a *completion* of a ranked space R, if R^* is complete and if there is a r-isomorphism of R into a r-dense ranked subspace of R^*.

The metric space R can be regarded as a ranked space. In fact, as a set $\mathfrak{B}_n$ $(n=0,1,2,\cdots)$ of neighbourhoods of rank n, if we take the set of all open spheres $S_{1/n+1}(p)$ of radius $1/n+1$ about p (p runs through the set R), R is a ranked space with indicator ω_0. Then, the completion in the ordinary sense of the metric space R is a completion of the ranked space R in the above sense.

5. Completion of the ranked space $\mathcal{E}$. First of all, let us consider the set $\mathcal{M}$ of all real valued measurable functions on $[a, b]$, and we regard two functions equal if they differ only in a set of measure zero. Let us introduce, as in $\mathcal{E}$, on the set $\mathcal{M}$ a set of neighbourhoods in the following way:

Definition 3. Given a closed subset A of $[a, b]$ and a positive number ε, the neighbourhood $V(A, \varepsilon\,;f)$, or simply $V(f)$, of the point f of $\mathcal{M}$ is the set of all those measurable functions $g(x)$ which are expressible as the sums of $f(x)$ and the other functions $r(x)$ with the following three properties:

$[\alpha]$ $|r(x)|<\varepsilon$ for all $x \in A$,

$[\beta]$ $k \operatorname{mes}\{x\,;|r(x)|>k\}<\varepsilon$ for each $k>0$,

$[\gamma]$ $\left|\int_a^b [r(x)]^k dx\right|<\varepsilon$ for each $k>0$.

Then, the neighbourhoods satisfy the axioms (A) and (B) of Hausdorff. The neighbourhoods $V(A, \varepsilon\,; f)$ and $V(B, \varepsilon\,; f)$ are identical if $\operatorname{mes}((A\backslash B)\cup(B\backslash A))=0$.

First, we obtain the following Lemma as in I,[3] Lemma 1.

Lemma 11. *If* $V(A, \varepsilon\,; f)\supseteq V(B, \eta\,; g)$, *then we have* $\operatorname{mes}(A\backslash B)=0$ *and* $\varepsilon \geq \eta$.

From this, we see that $\mathcal{M}$ is a space of depth ω_0. Hence, the indicator should be ω_0. For $n=0, 1, 2, \cdots$, a neighbourhood $V(A, \varepsilon\,; f)$ is said to be rank n, if it satisfies the condition

$[\delta]$ $\operatorname{mes}([a, b]\backslash A)<\varepsilon$ and $\varepsilon=2^{-n}$.

Then:

Proposition 4. *$\mathcal{M}$ is a ranked space of depth ω_0.*

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2) For this notion, see [1], [4], and [7].

3) The reference number indicates the number of the Note.

We have $\operatorname{mes}(A_n \setminus (\bigcap_{m=n}^{\infty} A_m)) = 0$ for the fundamental sequence $\{V(A_n, \varepsilon_n; f_n)\}$. Therefore, without loss of generality, we can always assume that $\{A_n\}$ is a monotone increasing sequence. Without notice, we consider, from now onwards, that it is always the case.

Lemma 12. *Let $\{f_n\}$ be a r-converging sequence of points in $\mathcal{M}$, then the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists and $\{\lim_n f_n\}$ is the set consisting of f alone.*

Proof. By the assumption, there is a function $f \in \mathcal{M}$ such that $f \in \{\lim_n f_n\}$, and so there is a fundamental sequence $\{V(A_n, \varepsilon_n; f)\}$ with $V(A_n, \varepsilon_n; f) \ni f_n$. We have then $|f(x) - f_n(x)| < \varepsilon_n$ for all $x \in A_n$. Hence, $\lim_{n\to\infty} f_n(x)$ exists and f coincides with the limit function.

Lemma 13. *If $f \in \{\lim_n f_n\}$ and $g \in \{\lim_n g_n\}$, then we have $\alpha f + \beta g \in \{\lim_n (\alpha f_n + \beta g_n)\}$ for any pair, α and β, of real numbers.*

Proof. By the assumptions, there are a fundamental sequence $\{V(A_n, \varepsilon_n; f)\}$ with $V(A_n, \varepsilon_n; f) \ni f_n$ and a fundamental sequence $\{V(B_n, \eta_n; g)\}$ with $V(B_n, \eta_n; g) \ni g_n$. Then, as it is easily seen, the sequence $\{V(A_n \cap B_n, \kappa_n; f+g)\}$, where $\kappa_n = 8 \max(\varepsilon_n, \eta_n)$, is fundamental. Moreover, we have $V(A_n \cap B_n, \kappa_n; f+g) \ni f_n + g_n$: in fact, for the function $(f_n + g_n) - (f+g)$, $[\alpha]$ and $[\beta]$ are obvious, $[\gamma]$ results by using II, Lemma 5. Thus, we obtain $f + g \in \{\lim_n (f_n + g_n)\}$. The sequence $\{V(A_n, 2^l \varepsilon_n; \alpha f)\}$, where l is the smallest positive integer such that $2^l \geq |\alpha|$, is fundamental, and it satisfies $V(A_n, 2^l \varepsilon_n; \alpha f) \ni \alpha f_n$. Hence, $\alpha f \in \{\lim_n \alpha f_n\}$.

Proposition 5. $\boldsymbol{K} = Cl_r(\mathcal{E})$.

Proof. Let $f \in \boldsymbol{K}$, then there is a fundamental sequence $\{V(A_n, \varepsilon_n; f_n)\}$ in $\mathcal{E}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$. By I, Lemma 3, $\{V(A_n, 2\varepsilon_n; f)\}$ in $\mathcal{M}$ is a fundamental sequence with $V(A_n, 2\varepsilon_n; f) \ni f_n$. Therefore, we have $f \in \{\lim_n f_n\}$. It shows that $f \in Cl_r(\mathcal{E})$. Let $f \in Cl_r(\mathcal{E})$, then there is a sequence $\{f_n\}$ of points of $\mathcal{E}$ r-converging to f in $\mathcal{M}$, so that there is a fundamental sequence $\{V(A_n, \varepsilon_n; f)\}$ in $\mathcal{M}$ with $V(A_n, \varepsilon_n; f) \ni f_n$. Let $\{n_i; i = 0, 1, 2, \cdots\}$ be an index sequence which satisfies the relation $\varepsilon_{n_i} \geq 2^6 \varepsilon_{n_{i+1}}$ for each i. Let us consider the sequence $\{V(A_i^*, \varepsilon_i^*; f_i^*)\}$ in $\mathcal{E}$ such that $A_{2i}^* = A_{2i+1}^* = A_{n_i}$, $\varepsilon_{2i}^* = 16\varepsilon_{n_i}$, $\varepsilon_{2i+1}^* = 8\varepsilon_{n_i}$ and $f_{2i}^* = f_{2i+1}^* = f_{n_i}$. Then $\{V(A_i^*, \varepsilon_i^*; f_i^*)\}$ is a fundamental sequence in $\mathcal{E}$, and we have, by Lemma 12, $\lim_{i\to\infty} f_i^*(x) = f(x)$. Thus, we obtain $f \in \boldsymbol{K}$.

Since $\mathcal{E}$ is a vector space, we see, by Lemma 13 and Proposition

5, that:

Proposition 6. *K is a vector space.*

We now introduce on K the neighbourhoods and the rank induced from $\mathcal{M}$. Then, K is a ranked subspace of $\mathcal{M}$. We also see that:

Proposition 7. *$\mathcal{E}$ is a ranked subspace of the ranked space K.*

Lemma 14. *Let $\{V(A_n, \varepsilon_n; f_n)\}$ be a fundamental sequence in K, then $f_n(x)$ converges to a finite function $f(x)$, and the integrals $I(f_n)$ converges to a finite limit.*

Proof. As in I, Lemma 2, we can prove the convergence of $f_n(x)$. We have $|I(f_n)-I(f_m)| \leq \left|I(f_n)-\int_a^b [f_n(x)]^k dx\right| + \left|\int_a^b [f_n(x)]^k dx - \int_a^b [f_m(x)]^k dx\right| + \left|\int_a^b [f_m(x)]^k dx - I(f_m)\right|$. The second term can be estimated, by II, Lemma 5, as follows: $\left|\int_a^b [f_n(x)]^k dx - \int_a^b [f_m(x)]^k dx\right| \leq \left|\int_a^b [f_m(x) - f_n(x)]^{2k} dx\right| + 2k[\text{mes}\{x; |f_n(x)|>k\} + \text{mes}\{x; |f_m(x)|>k\}]$. Hence, the convergence of $I(f_n)$ follows from II, Lemma 9.

Lemma 15. *Let $\{V(A_n, \varepsilon_n; f_n)\}$ be a fundamental sequence in K, and put $f(x)=\lim_{n\to\infty} f_n(x)$. Then, $f \in K$ and $\bigcap_{n=0}^{\infty} V_n(f_n)=\{f\}$.*

Proof. As in I, Lemma 3, first we have, for each n,

i) $|f(x)-f_n(x)| \leq \varepsilon_n$ for all $x \in A_n$,

ii) $k\,\text{mes}\{x; |f(x)-f_n(x)|>k\} \leq \varepsilon_n$ for each $k>0$,

iii) $\left|\int_a^b [f(x)-f_n(x)]^k dx\right| \leq \varepsilon_n$ for each $k>0$.

Since we have k mes $\{x; |f(x)|>k\} \leq 2[k/2\,\text{mes}\{x; |f(x)-f_n(x)|>k/2\} + k/2\text{mes}\{x; |f_n(x)|>k/2\}]$ and since $f_n \in K$, $\lim_{n\to\infty} k\,\text{mes}\{x; |f(x)|>k\}=0$ follows from ii) and II, Lemma 9. Moreover, we have $\left|\int_a^b [f(x)]^k dx - I(f_n)\right| \leq \left|\int_a^b [f(x)]^k dx - \int_a^b [f_n(x)]^k dx\right| + \left|\int_a^b [f_n(x)]^k dx - I(f_n)\right|$. The first term can be estimated, by II, Lemma 5, as follows: $\left|\int_a^b [f(x)]^k dx - \int_a^b [f_n(x)]^k dx\right| \leq \left|\int_a^b [f(x)-f_n(x)]^{2k} dx\right| + 2[k\,\text{mes}\{x; |f(x)|>k\} + k\,\text{mes}\{x; |f_n(x)|>k]$. Therefore, $\lim_{k\to\infty} \int_a^b [f(x)]^k dx$ exists by iii), II, Lemma 9 and Lemma 14. Thus, $f \in K$ results from II, Lemma 10. On the other hand, from that $\{V_n(f_n)\}$ is a fundamental sequence, there is a sub-sequence $\{V_{n_i}(f_{n_i}); i=0, 1, 2, \cdots\}$ such that $f_{n_{2i}}=f_{n_{2i+1}}$ and $\varepsilon_{n_{2i}} > \varepsilon_{n_{2i+1}}$ $(i=0, 1, 2, \cdots)$. Hence, $f \in V_{n_{2i}}(f_{n_{2i}})$ for each i, so that $f \in \bigcap_{n=0}^{\infty} V_n(f_n)$. Suppose that g is a point belonging to $\bigcap_{n=0}^{\infty} V_n(f_n)$, then we have $|g(x)-f_n(x)|<\varepsilon_n$ for all $x \in A_n$, and so g coincides with f.

Lemma 15 asserts that:

Proposition 8. *K is a complete ranked space.*

Theorem 3. *K is a completion of $\mathcal{E}$.*

Proof. By Proposition 5, for each $f \in \mathcal{M}$, there is a sequence $\{f_n\}$ of points of $\mathcal{E}$ r-converging to f in $\mathcal{M}$. Then, the sequence $\{f_n\}$ r-converges to f in K. Therefore, $\mathcal{E}$ is r-dense in K. Thus, from Propositions 7 and 8, the desired assertion follows.

6. Characterization of the (E.R.) integral in the special sense. As it is easily seen from the definition, we first have the following Proposition.

Proposition 9. *If $f \in \mathcal{E}$, $I(f)$ coincides with the integral $\int_a^b f(x)dx$ defined for f as usual.*

Proposition 10. *If $f \in \{\lim_n f_n\}$ in K, then $\lim_{n\to\infty} I(f_n)=I(f)$.*

Proof. This is proved by use of the same method as Lemma 14.

Propositions 5, 9, and 10 assert that the integral I is the r-continuous extension of integrals of step functions.

Proposition 11. *If $f \in K$ and $g \in K$, then, for any pair, α and β, of real numbers, we have $I(\alpha f+\beta g)=\alpha I(f)+\beta I(g)$.*

Proof. From that $I(f)$, $f \in K$, is the r-continuous extension of integrals of step functions, and that $I(f)$ is a linear functional on $\mathcal{E}$, our assertion results by Lemma 13.

Propositions 10 and 11 assert that:

Theorem 4. *$I(f)$ is a r-continuous linear functional on K.*

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