204. On the Product of M-Spaces. II

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1. This is the continuation of our previous paper [1].*) The purpose of this paper is to prove the following theorems which are related to the product of M-spaces and to the countable product of the spaces belonging to $\mathfrak{C}$.

Theorem 1.1. *If X belongs to $\mathfrak{C}$, then the product $X\times Y$ is an M-space for any M-space Y.*

Corollary 1.2. *If X is an M-space which satisfies one of the following conditions, then the product $X\times Y$ is also an M-space for any M-space Y.*

(a) *X satisfies the first axiom of countability.*

(b) *X is locally compact.*

(c) *X is paracompact.*

Since an M-space X which satisfies one of conditions (a), (b), and (c) belongs to $\mathfrak{C}$ by [1, Theorem 2.2], this corollary is a direct consequence of Theorem 1.1.

Theorem 1.3. *If $X_n, n=1, 2, \cdots$, are the spaces belonging to $\mathfrak{C}$, then the product $\prod_{n=1}^{\infty} X_n$ also belongs to $\mathfrak{C}$.*

Corollary 1.4. *If $X_n, n=1, 2, \cdots$, are M-spaces each of which satisfies the first axiom of countability, then the product $\prod_{n=1}^{\infty} X_n$ is also an M-space satisfying the first axiom of countability.*

If each space X_n satisfies the first axiom of countability, then the product $\prod_{n=1}^{\infty} X_n$ satisfies the first axiom of countability, too. Hence this corollary follows from Theorem 1.3 directly.

If each space X_n is a paracompact M-space, then the product $\prod_{n=1}^{\infty} X_n$ is also a paracompact M-space (cf. K. Morita [3, Theorem 6.4]). However for locally compact M-spaces X_n, the product $\prod_{n=1}^{\infty} X_n$ is not locally compact in general. For example, let $X_n, n=1, 2, \cdots$, be the spaces of real numbers with the usual topology. Then the product $\prod_{n=1}^{\infty} X_n$

*) All spaces are assumed to be Hausdorff.

belongs to $\mathfrak{C}$, while it is not locally compact (cf. [2, Theorem 19 in Chap. 5]).

2. Lemmas. Lemma 2.1. *Let $\{\mathfrak{U}_i\}$ and $\{\mathfrak{V}_i\}$ be normal sequences of open coverings of the spaces X and Y, respectively. If we put $\mathfrak{W}_i=\{U\times V \mid U\in\mathfrak{U}_i,\ V\in\mathfrak{V}_i\}$ for each i, then $\{\mathfrak{W}_i\}$ is a normal sequence of open coverings of the product $X\times Y$.*

Proof. Let $W=U\times V\in\mathfrak{W}_{i+1}$, where $U\in\mathfrak{U}_{i+1}$ and $V\in\mathfrak{V}_{i+1}$. Then $\mathrm{St}(W, \mathfrak{W}_{i+1})=\mathrm{St}(U, \mathfrak{U}_{i+1})\times\mathrm{St}(V, \mathfrak{V}_{i+1})$. Since $\mathrm{St}(U, \mathfrak{U}_{i+1})\subset U'$ and $\mathrm{St}(V, \mathfrak{V}_{i+1})\subset V'$ for some $U'\in\mathfrak{U}_i$ and for some $V'\in\mathfrak{V}_i$, we have $\mathrm{St}(W, \mathfrak{W}_{i+1})\subset U'\times V'\in\mathfrak{W}_i$, which shows that $\{\mathfrak{W}_i\}$ is a normal sequence of open coverings of $X\times Y$.

Lemma 2.2. *For each positive integer n, let $\{\mathfrak{U}(n, i)\mid i=1, 2, \cdots\}$ be a normal sequence of open coverings of a space X_n. If we put*

$$\mathfrak{U}_i=\{U_1\times\cdots\times U_i\times\prod_{n>i}X_n \mid U_j\in\mathfrak{U}(j, i),\ j=1, \cdots, i\},$$

then $\{\mathfrak{U}_i\}$ is a normal sequence of open coverings of the product $\prod_{n=1}^{\infty}X_n$.

Proof. Let $V=U_1\times\cdots\times U_{i+1}\times\prod_{n>i+1}X_n\in\mathfrak{U}_{i+1}$, where $U_j\in\mathfrak{U}(j, i+1)$, $j=1, \cdots, i+1$. Then we have

$$\mathrm{St}(V, \mathfrak{U}_{i+1})=\mathrm{St}(U_1, \mathfrak{U}(1, i+1))\times\cdots\times\mathrm{St}(U_{i+1}, \mathfrak{U}(i+1, i+1))\times\prod_{n>i+1}X_n.$$

Since $\mathrm{St}(U_j, \mathfrak{U}(j, i+1))\subset U'_j$ for some $U'_j\in\mathfrak{U}(j, i)$, $\mathrm{St}(V, \mathfrak{U}_{i+1})$ is contained in $U'_1\times\cdots\times U'_i\times\prod_{n>i}X_n\in\mathfrak{U}_i$. Hence $\mathfrak{U}_{i+1}$ is a star refinement of $\mathfrak{U}_i$ for each i. Thus we complete the proof.

Lemma 2.3. *If X is a compact space, and if Y is a countably compact space, then the product $X\times Y$ is countably compact.*

This lemma is due to J. Novák [4, Theorem 5].

3. Proof of Theorem 1.1. Let $\{\mathfrak{U}_i\}$ be a normal sequence of open coverings of X satisfying Condition $(*)$, and let $\{\mathfrak{V}_i\}$ be a normal sequence of open coverings of Y satisfying Condition (M_0). If we put $\mathfrak{W}_i=\{U\times V\mid U\in\mathfrak{U}_i,\ V\in\mathfrak{V}_i\}$ for each i, then by Lemma 2.1 $\{\mathfrak{W}_i\}$ is a normal sequence of open coverings of $X\times Y$. Let $\{z_i\}$ be a sequence of points of $X\times Y$ such that $z_i\in\mathrm{St}(z_0, \mathfrak{W}_i)$ for each i and for some fixed point z_0 of $X\times Y$. Let us put $z_i=(x_i, y_i)\in X\times Y$ and $z_0=(x_0, y_0)\in X\times Y$. Since $x_i\in\mathrm{St}(x_0, \mathfrak{U}_i)$ for each i, there exists a subsequence $\{x_{i(n)}\}$ of $\{x_i\}$ which has the compact closure in X. On the other hand, since $y_i\in\mathrm{St}(y_0, \mathfrak{V}_i)$ for each i, any subsequence of $\{y_i\}$ has an accumulation point in $\bigcap\mathrm{St}(y_0, \mathfrak{V}_i)$ and nowhere else, which shows that the closure of $\{y_i\}$ in Y is countably compact. Hence the closure of $\{y_{i(n)}\}$ in Y is also countably compact. Consequently, by Lemma 2.3 $\overline{\{x_{i(n)}\}}\times\overline{\{y_{i(n)}\}}$ is countably compact. This shows that $\{z_{i(n)}\}$ has an accumulation point in $X\times Y$, and hence $X\times Y$ is an M-space. Thus we complete the proof.

Proof of Theorem 1.3. Let us put $X=\prod_{n=1}^{\infty} X_n$, and let $\{\mathfrak{U}(n, i) \mid i=1, 2, \cdots\}$ be a normal sequence of open coverings of X_n satisfying Condition (*). Then, as is shown in Lemma 2.2, we can construct a normal sequence $\{\mathfrak{U}_i\}$ of open coverings of X. Now let $\{x(i) \mid i=1, 2, \cdots\}$ be a sequence of points of X such that $x(i) \in \mathrm{St}(x, \mathfrak{U}_i)$ for each i and for some fixed point x of X. We denote the k-th coordinate of a point x of X by x_k. If we put

$$x(i)=(x_1(i), x_2(i), \cdots, x_k(i), \cdots), \ x_k(i) \in X_k,$$

then $\{x_1(i) \mid i=1, 2, \cdots\}$ is a sequence of points of X_1 such that $x_1(i) \in \mathrm{St}(x_1, \mathfrak{U}(1, i))$. Hence by Condition (*) there exists a subsequence $\{x_1(n_{1i}) \mid i=1, 2, \cdots\}$ of $\{x_1(i)\}$ which has the compact closure in X_1, where we may assume that $2 \leqq n_{1i} < n_{1,i+1}$, $i=1, 2, \cdots$. Next we consider a sequence $\{x_2(n_{1i}) \mid i=1, 2, \cdots\}$ of points of X_2. Since $x_2(n_{1i}) \in \mathrm{St}(x_2, \mathfrak{U}(2, i))$ for each i, there exists a subsequence $\{x_2(n_{2i}) \mid i=1, 2, \cdots\}$ of $\{x_2(n_{1i})\}$ which has the compact closure in X_2, where we may assume that $3 \leqq n_{2i} < n_{2,i+1}$, $i=1, 2, \cdots$. By repeating these processes, we can select a subsequence $\{x_k(n_{ki}) \mid i=1, 2, \cdots\}$ of $\{x_k(n_{k-1,i})\}$ which has the compact closure in X_k for each $k \geqq 2$, where we may assume that $k+1 \leqq n_{ki} < n_{k,i+1}$, $i=1, 2, \cdots$. Now consider the subsequence $\{x(n_{kk}) \mid k=1, 2, \cdots\}$ of $\{x(i) \mid i=1, 2, \cdots\}$. Then we can prove that the closure of $\{x(n_{kk}) \mid k=1, 2, \cdots\}$ in X is compact. In fact, if we put

$$K_1=\overline{\{x_1(n_{1i})\}}, \ K_k=\overline{\{x_k(n_{ki})\}} \cup \{x_k(n_{1i}) \mid n_{1i} < n_{kk}\}, \qquad k=2, 3, \cdots,$$

then $\{x(n_{kk}) \mid k=1, 2, \cdots\}$ is contained in a compact set $K=\prod_{k=1}^{\infty} K_k$. Since K is compact in X, the closure of $\{x(n_{kk}) \mid k=1, 2, \cdots\}$ in X is compact. This completes the proof.

References

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