232. On M- and M*-Spaces

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1. In [2], K. Morita has introduced the notion of M-spaces. A topological space X is an M-space if there exists a normal sequence $\{\mathfrak{A}_i \mid i=1,2,\cdots\}$ of open coverings of X satisfying the condition (M) below:

(M) If $\{K_i\}$ is a sequence of non-empty subsets of X such that $K_{i+1}\subset K_i$, $K_i\subset \mathrm{St}(x_0, \mathfrak{A}_i)$ for each i and for some fixed point x_0 of X, then $\bar{K}_i \neq \phi$.

On the other hand, in [1], we introduced the notion of M^*-spaces. A topological space X is an M^*-space if there exists a sequence $\{\mathfrak{F}_i \mid i=1,2,\cdots\}$ of locally finite closed coverings of X satisfying Condition (M), where we may assume without loss of generality that $\mathfrak{F}_{i+1}$ is a refinement of $\mathfrak{F}_i$ for each i. As for the relations between M- and M^*-spaces, the following results are proved by K. Morita [3].

(1) There exists an M^*-space which is locally compact Hausdorff but is not an M-space.

(2) A collectionwise normal space is an M-space if and only if it is an M^*-space.

The first result is a direct consequence of the following (cf. [3]): There is a perfect map $f: X\rightarrow Y$ such that X is an M-space but Y is not, and such that X, Y are locally compact Hausdorff spaces. In fact, the space Y is an M^*-space as the image under a perfect map f of an M^*-space X by [1, Theorem 2.3 in I].[1] However, it seems to be unknown whether a normal M^*-space is an M-space or not. The purpose of this paper is to give an affirmative answer for this problem.

2. We shall prove the following main theorem.

Theorem 2.1. *A normal space X is an M-space if and only if it is an M^*-space.*

Before proving Theorem 2.1, we mention a fundamental lemma, which is essentially due to K. Morita [3].

Lemma 2.2. *Let X be an M^*-space with a sequence $\{\mathfrak{F}_i\}$ of locally finite closed coverings of X satisfying Condition* (M), *where $\mathfrak{F}_{i+1}$ is a refinement of $\mathfrak{F}_i$ for each i. Then the following statements are valid.*

(a) *If $\{K_i\}$ is a sequence of non-empty subsets of X such that*

1) In [1, Theorem 2.3 in I], the assumption that X is T_1 is unnecessary.

$K_{i+1}\subset K_i$, $K_i\subset \mathrm{St}^k(x_0,\mathfrak{F}_i)$, $i=1,2,\cdots$, *for some fixed positive integer* k *and for some fixed point* x_0 *of* X, *then* $\cap \bar{K}_i\neq\phi$, *where* $\mathrm{St}^k(x_0,\mathfrak{F}_i)$ *denotes the* k*-times iterated star of a point* x_0 *in each covering* $\mathfrak{F}_i$.

(b) *Let* $\mathfrak{W}_i=\{\mathrm{St}(F,\mathfrak{F}_i)\,|\,F\in\mathfrak{F}_i\}$ *for each* i. *Then the sequence* $\{\mathfrak{W}_i\}$ *of the coverings of* X *satisfies Condition* (M).

Proof of Theorem 2.1. Since any M-space is clearly an M^*-space, we shall prove only that a normal M^*-space X is an M-space. Let $\{\mathfrak{F}_i\,|\,i=1,2,\cdots\}$ be a sequence of locally finite closed coverings of X satisfying Condition (M), where we may assume that $\mathfrak{F}_{i+1}$ is a refinement of $\mathfrak{F}_i$ for each i. If we put $C(x)=\cap\{\mathrm{St}^2(x,\mathfrak{F}_i)\,|\,i=1,2,,\cdots\}$ for any point x of X, then the set $C(x)$ is countably compact by Lemma 2.2 (a). As is easily shown, if $\mathfrak{F}$ is any locally finite collection of the subsets of X, then a countably compact subset C of X intersects with only finite members of $\mathfrak{F}$. Hence the set $C(x)$ intersects with only finite members of each $\mathfrak{F}_n$. Let us put

$$U_n(x)=X-\cup\{F\,|\,F\cap C(x)=\phi,\quad F\in\mathfrak{F}_n\},\quad n=1,2,\cdots$$

for any point x of X. Then each $U_n(x)$ is open in X, and $C(x)\subset U_n(x)$. Therefore it follows from Lemma 2.2 (a) that for each n there exists some positive integer $k(n)$ such that $\mathrm{St}^2(x,\mathfrak{F}_{k(n)})\subset U_n(x)$. Since $\mathrm{St}^2(x,\mathfrak{F}_{k(n)})$ intersects with only finite members of $\mathfrak{F}_n$, $\mathrm{St}(x,\mathfrak{F}_{k(n)})$ intersects with only finite members of $\{\mathrm{St}(F,\mathfrak{F}_{k(n)})\,|\,F\in\mathfrak{F}_n\}$. Now for each n and k, let us denote by G_{nk} the subset of X which consists of points x of X such that $\mathrm{St}(x,\mathfrak{F}_k)$ intersects with only finite members of $\{\mathrm{St}(F,\mathfrak{F}_k)\,|\,F\in\mathfrak{F}_n\}$. Then each G_{nk} is an open subset of X. Indeed, let $x\in G_{nk}$, and put

$$H_k(x)=X-\cup\{F\,|\,x\notin F, F\in\mathfrak{F}_k\}.$$

Then $H_k(x)$ is open in X, and if $y\in H_k(x)$, then $\mathrm{St}(y,\mathfrak{F}_k)\subset\mathrm{St}(x,\mathfrak{F}_k)$, which implies that $y\in G_{nk}$. Hence each G_{nk} is open in X. Further it follows easily that $G_{nk}\subset G_{n,k+1}$, $k=1,2,\cdots$, and that $\{G_{nk}\,|\,k=1,2,\cdots\}$ is a covering of X for each n. On the other hand, a normal M^*-space X is countably paracompact by [1, Theorem 2.7 in I]. Therefore for each n there exists a locally finite open refinement $\{H_{nk}\,|\,k=1,2,\cdots\}$ of a countable open covering $\{G_{nk}\,|\,k=1,2,\cdots\}$ of X such that $\bar{H}_{nk}\subset G_{nk}$, $k=1,2,\cdots$. Let us put

$$\mathfrak{H}(n,k)=\{H_{nk}\cap\mathrm{Int}(\mathrm{St}(F,\mathfrak{F}_{\max(n,k)}))\,|\,F\in\mathfrak{F}_n\},$$
$$\mathfrak{H}(n)=\cup\{\mathfrak{H}(n,k)\,|\,k=1,2,\cdots\},$$

for each n and k. Then each $\mathfrak{H}(n,k)$ is a locally finite collection of open subsets of X. Indeed, let $x\in\bar{H}_{nk}$. Since $\bar{H}_{nk}\subset G_{n,\max(n,k)}$, $\mathrm{St}(x,\mathfrak{F}_{\max(n,k)})$ intersects with only finite members of $\{\mathrm{St}(F,\mathfrak{F}_{\max(n,k)})\,|\,F\in\mathfrak{F}_n\}$, and hence $\mathrm{Int}(\mathrm{St}(x,\mathfrak{F}_{\max(n,k)}))$ intersects with only finite members of $\{\mathrm{Int}(\mathrm{St}(F,\mathfrak{F}_{\max(n,k)}))\,|\,F\in\mathfrak{F}_n\}$. This shows that $\mathfrak{H}(n,k)$ is a locally finite collection of open subsets of X for each n and k. Therefore

$\mathfrak{H}(n)$ is a locally finite open covering of X for each n. Let us put

$$\mathfrak{A}_n=\{\mathrm{Int}(\mathrm{St}(F,\mathfrak{F}_n)\,|\,F\in\mathfrak{F}_n\},\qquad n=1,2,\cdots.$$

Then by Lemma 2.2 (b), the sequence $\{\mathfrak{A}_n\,|\,n=1,2,\cdots\}$ of open coverings of X satisfies Condition (M). Further, since $n\leqq\max(n,k)$, it follows that each locally finite open covering $\mathfrak{H}(n)$ of X refines $\mathfrak{A}_n$. Since X is a normal space, any locally finite open covering of X is a normal covering (cf. [4]), and hence each $\mathfrak{A}_n$ is a normal open covering of X. Therefore for each n there exists a normal sequence

$$\mathfrak{A}_n>{}^*\mathfrak{A}_{n1}>{}^*\mathfrak{A}_{n2}>{}^*\cdots>{}^*\mathfrak{A}_{ni}>{}^*>\cdots$$

of open coverings of X. Let us put

$$\mathfrak{B}_1=\mathfrak{A}_1,\quad \mathfrak{B}_i=\mathfrak{A}_i\cap\mathfrak{A}_{1,i-1}\cap\cdots\cap\mathfrak{A}_{i-1,1},\quad i=2,3,\cdots.$$

Then $\{\mathfrak{B}_i\,|\,i=1,2,\cdots\}$ is a normal sequence of open coverings of X satisfying Condition (M). Hence X is an M-space. Thus we complete the proof of Theorem 2.1.

3. Let X and Y be topological spaces. A map $f:X\to Y$ is called a quasi-perfect map if it is a continuous surjective map such that $f^{-1}(y)$ is countably compact for each point y of Y (cf. [3]). In [1], we proved that, if $f:X\to Y$ is a quasi-perfect map and if X is an M^*-space, then Y is also an M^*-space. Combining this result with our main theorem, we can obtain the following

Theorem 3.1 (cf. [1, Theorem 1.1 in II], [3, Theorem 2.2]). *Let $f:X\to Y$ be a quasi-perfect map. If X is an M-space and if either X or Y is normal, then Y is an M-space.*

References

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