10. On Weak Convergence of Transformations in Topological Measure Spaces

By Ryotaro SATO
Department of Mathematics, Josai University, Saitama

(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 13, 1969)

1. Introduction. A sequence $\{T_n\}$ of invertible measure-preserving transformations in the unit interval [0, 1] is said to be convergent weakly to the invertible measure-preserving transformation T if $\lim_{n\to\infty} \|f\circ T_n - f\circ T\| = 0$ for every integrable function f, with $\|\cdot\|$ denoting L^1-norm. It is well-known that (α) and (β) in Theorem 1 below are equivalent.

In this paper we prove that if X is a locally compact metrizable space and μ a σ-finite Radon measure on X, then the equivalence between (α) and (β) also holds (Theorem 1). We see that this generalizes a theorem of Papangelou [2, Theorem 2]. Then it will be natural to ask: does the metrizability of X be dropped in Theorem 1 when X is a compact Hausdorff space? Theorem 3 asserts that the answer is negative.

2. An extension of Papangelou's theorems. Let X be a locally compact Hausdorff space and $\mathfrak{B}$ the σ-field generated by the open subsets of X. The members of $\mathfrak{B}$ will be called the Borel subsets of X. Let μ_1 be a measure on $\mathfrak{B}$ such that

(i) $\mu_1(K)$ is finite for every compact subset K of X,

(ii) $\mu_1(V) = \sup\{\mu_1(K) \mid K$ is compact and $K \subset V\}$ for every open subset V of X,

(iii) $\mu_1(A) = \inf\{\mu_1(V) \mid V$ is open and $A \subset V\}$ for every Borel subset A of X.

We denote by μ the outer measure induced by μ_1 and denote by $\mathfrak{M}$ the σ-field of all subsets of X which are μ-measurable. We say μ on $\mathfrak{M}$ a Radon measure on X. A subset E of X which belongs to $\mathfrak{M}$ will be called measurable in X.

We denote by G the group of all invertible μ-measure-preserving transformations in X.

Difinition. The sequence $\{T_n\}$ in G converges to $T \in G$ weakly if $\lim_{n\to\infty} \mu(T_nA + TA) = 0$ for every measurable subset A of X with $\mu(A) < \infty$, or equivalently, if $\lim_{n\to\infty} \|f\circ T_n - f\circ T\| = 0$ for every $f \in L^1$.

Theorem 1. *Let X be a locally compact metrizable space and μ a σ-finite Radon measure on X. If T, T_n ($n = 1, 2, 3, \cdots$) are in G then*

(α) and (β) below are equivalent:

(α) *$\{T_n\}$ converges to T weakly.*

(β) *Every subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ has a subsequence $\{T_{k(l(n))}\}$ which converges to T almost everywhere.*

The proof of Theorem 1 requires some lemmas.

Lemma 1. *Let μ be a σ-finite Radon measure on a locally compact Hausdorff space X. Then there exists a σ-compact set E such that $\mu(X-E^\circ)=0$, where E° is the interior of E.*

Proof. Let $X=\bigcup_{n=1}^{\infty} X_n$ and X_n $(n=1, 2, 3, \cdots)$ be mutually disjoint Borel sets with finite measure. Let ε be an arbitrary positive rational number. By the property (iii) of μ, there exists an open set V_n in X such that $\mu(V_n-X_n)<\varepsilon/2^{n+1}$ and $X_n\subset V_n$. Then by the property (ii) of μ, there exists a compact set K_n in X such that $\mu(V_n-K_n)<\varepsilon/2^{n+1}$ and $K_n\subset V_n$. Hence we have

$$\mu(K_n+X_n)\leqq\mu(K_n+V_n)+\mu(V_n+X_n)<\varepsilon/2^{n+1}+\varepsilon/2^{n+1}=\varepsilon/2^n.$$

Therefore

$$\mu(X-\cup K_n)\leqq\sum_{n=1}^{\infty}\mu(K_n+X_n)<\sum_{n=1}^{\infty}\varepsilon/2^n=\varepsilon.$$

Now if we choose a compact set $K_n(\varepsilon)$ such that $(K_n(\varepsilon))^\circ\supset K_n$, and if we put

$$E=\cup\{K_n(\varepsilon)\,|\,n=1, 2, 3, \cdots;\ \varepsilon \text{ is a positive rational number}\}$$

then E is σ-compact and $\mu(X-E^\circ)=0$. The proof is completed.

Let T_1 and T_2 be mappings of X into itself. Then we define the mapping denoted by $T_1\times T_2$ of X into $X\times X$ as follows:

$$(T_1\times T_2)x=(T_1x, T_2x) \qquad (x\in X).$$

Lemma 2. *Let X be a locally compact metrizable space and μ a σ-finite Radon measure on X. If T_1 and T_2 are measure-preserving transformations of $(X, \mathfrak{M}, \mu)$ into itself, then the inverse image $(T_1\times T_2)^{-1}(B)$ of every Borel subset B of $X\times X$ is a measurable subset of X.*

Proof. For the proof if it sufficient to show that $(T_1\times T_2)^{-1}(V)$ is a measurable subset of X for any open subset V of $X\times X$. Let V be open in $X\times X$. By Lemma 1 there exists a σ-compact subset E of X such that $\mu(X-E)=0$. Evidently E is separable. Let $\{x_n\,|\,n=1,2,3,\cdots\}^-\supset E$, and put $F=\{x_n\,|\,n=1, 2, 3, \cdots\}^-$. Let d be a metric on X which is compatible with the topology of X. Then we have

$$V\cap(F\times F)$$
$$\subset\bigcup\left\{U(x_n)\times U(x_m)\,\middle|\,\begin{array}{l}U(x_n),\ U(X_m) \text{ are some } \varepsilon\text{-neighborhoods of } x_n,\\ x_m, \text{ respectively, where } \varepsilon \text{ is rational and}\\ U(x_n)\times U(x_m)\subset V\end{array}\right\}.$$

In fact, if $(x, y)\in V\cap(F\times F)$ then there exist ε-neighborhoods $U(x)$ and $U(y)$ such that $U(x)\times U(y)\subset V$. Since $\{x_n\,|\,n=1, 2, 3, \cdots\}$ is dense in

F, then for some x_n and x_m it follows that $d(x, x_n) < \varepsilon/3$ and $d(y, x_m) < \varepsilon/3$. If $U(x_n)$ and $U(x_m)$ are $2\varepsilon/3$-neighborhoods of x_n and x_m, respectively, then $(x, y) \in U(x_n) \times U(x_m) \subset V$. Hence

$$\begin{aligned}(T_1\times T_2)^{-1}(V)\\ &=(T_1\times T_2)^{-1}(V-(F\times F))\cup(T_1\times T_2)^{-1}(V\cap(F\times F))\\ &=(T_1\times T_2)^{-1}(V-(F\times F))\cup(T_1\times T_2)^{-1}(\cup\{U(x_n)\times U(x_m)\})\\ &=(T_1\times T_2)^{-1}(V-(F\times F))\cup\bigcup\{(T_1\times T_2)^{-1}(U(x_n)\times U(x_m))\}. \qquad (1)\end{aligned}$$

On the other hand, $(T_1\times T_2)^{-1}(V-(F\times F))$ is contained in $T_1^{-1}(X-F) \cup T_2^{-1}(X-F)$ of measure zero and so it is measurable. The measurability of $(T_1\times T_2)^{-1}(U(x_n)\times U(x_m))$ is now obvious. By (1), $(T_1\times T_2)^{-1}(V)$ is a countable union of measurable subsets of X and hence it is measurable. This completes the proof.

Now using the above lemmas, we prove Theorem 1.

Proof of Theorem 1. (α) implies (β): By Lemma 1, there exists a σ-compact set $E=\cup\{K_n \mid n=1, 2, 3, \cdots\}$ such that K_n is compact for each n and $\mu(X-E^\circ)=0$. Since E is separable, there exists a countable set $\{x_n \mid n=1, 2, 3, \cdots\}$ in E such that $\{x_n \mid n=1, 2, 3, \cdots\}^- \supset E$. We put $F=\{x_n \mid n=1, 2, 3, \cdots\}^-$. Then $F^\circ \supset E^\circ$. Thus

$$\mu(X-F^\circ)=0. \qquad (2)$$

Let F_∞ be the one point compactification of F. Since F_∞ is a compact Hausdorff space with countable open basis, F_∞ is metrizable with some metric d. If we denote by $\mathfrak{C}(F_\infty)$ the space of all real-valued continuous functions on F_∞, then using the Stone-Weierstrass theorem it can be easily shown that $\mathfrak{C}(F_\infty)$ is separable relative to its uniform topology. Since the space $\mathfrak{C}_{00}(F)$ of all real-valued continuous functions on F with compact supports is a subspace of $\mathfrak{C}(F_\infty)$, $\mathfrak{C}_{00}(F)$ is separable relative to its uniform topology. Let $\{f_j \mid j=1, 2, 3, \cdots\}$ be a countable dense subset of $\mathfrak{C}_{00}(F)$. We extended f_j to g_j on X as follows: $g_j(x) = f_j(x)$ if $x \in F$ and $g_j=0$ on $X-F$. Then g_j is an integrable function on X. By (α), we have

$$\lim_{n\to\infty}\int_X |(g_j\circ T_n)x-(g_j\circ T)x|\,d\mu(x)=0$$

for $j=1, 2, 3, \cdots$. Thus for each j there exists a subsequence $\{T_{k(j,n)}\}$ of $\{T_n\}$ such that

$$\lim_{n\to\infty} g_j(T_{k(j,n)}x)=g_j(Tx) \qquad \text{a.e.} \qquad (3)$$

Therefore we can apply the Cantor diagonalization technique to obtain a subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ and a set N of measure zero such that if $x \notin N$

$$\lim_{n\to\infty} g_j(T_{k(n)}x)=g_j(Tx) \qquad \text{for each } j. \qquad (4)$$

Then we see that

$$\lim_{n\to\infty} T_{k(n)}x=Tx \qquad \text{a.e.} \qquad (5)$$

In fact, $N\cup T^{-1}(X-F^\circ)\cup\bigcup\{T_n^{-1}(X-F^\circ) \mid n=1, 2, 3, \cdots\}$ is of measure zero, and if $x \notin N \cup T^{-1}(X-F^{\circ}) \cup \bigcup\{T_n^{-1}(X-F^{\circ}) \mid n=1, 2, 3, \cdots\}$ then

$$Tx,\ T_n x \in F. \tag{6}$$

Let $V(Tx)$ be a neighborhood of Tx such that $V(Tx) \subset F^{\circ}$ and $V(Tx)^{-}$ is compact. Let h be a continuous function on X such that $0 \leqq h \leqq 1$, $h(Tx)=1$ and $h=0$ on $X-V(Tx)$. The restriction of h to F is a function of $\mathfrak{C}_{00}(F)$. Thus there exists an i_0 such that

$$|h(y)-f_{i_0}(y)| < 1/3 \qquad \text{for all } y \in F. \tag{7}$$

Since $x \notin N$,

$$\lim_{n\to\infty} g_{i_0}(T_{k(n)}x) = g_{i_0}(Tx).$$

Hence there exists some N_0 such that $n \geqq N_0$ implies $|g_{i_0}(T_{k(n)}x) - g_{i_0}(Tx)| < 1/3$. Comparing (6) and (7), it follows that if $n \geqq N_0$ then $|h(Tx) - f_{i_0}(T_{k(n)}x)| < 2/3$. Since $h(Tx)=1$, this implies that $f_{i_0}(T_{k(n)}x) > 1/3$ for $n \geqq N_0$. Then from (7),

$$h(T_{k(n)}x) > 0 \qquad \text{for each } n \geqq N_0. \tag{8}$$

This implies that $\{T_{k(n)}x\}$ converges to Tx.

(β) implies (α): By virtue of Lemma 2, the proof runs on the same line as that of corresponding part of [2, Theorem 2], and so we omit the proof here.

Theorem 2. *Let X be a locally compact metrizable space and μ a σ-finite Radon measure on X. Let G be the group of all automorphisms of the measure space $(X, \mathfrak{M}, \mu)$. The weak topology on G is the finest topology $\mathfrak{T}$ such that if a sequence $\{T_n\}$ in G converges to a transformation T in G almost everywhere then $\mathfrak{T}-\lim T_n = T$.*

Proof. A proof analogous to that of [2, Theorem 3] suffices.

3. A counter-example for a compact non-metrizable space. In this section we show that the equivalence between (α) and (β) in Theorem 1 does not necessarily hold when X is a compact non-metrizable Hausdorff space and μ a Radon measure on X.

Let A be any nonvoid index set and let for each α in A there correspond a compact abelian group H_α with the normalized Haar measure λ_α on $\mathfrak{M}_\alpha$, where $\mathfrak{M}_\alpha$ is the σ-field of the λ_α-measurable subsets of H_α. We denote by $(\otimes H_\alpha, \otimes\mathfrak{M}_\alpha, \otimes\lambda_\alpha)$ the product measure space of the measure spaces $(H_\alpha, \mathfrak{M}_\alpha, \lambda_\alpha)$. Then we have the following

Lemma 3. *The above $\otimes\lambda_\alpha$ is the restriction to $\otimes\mathfrak{M}_\alpha$ of the normalized Haar measure m on $H \equiv \otimes H_\alpha$ considered as the direct topological group of H_α. Moreover the outer measure induced by $\otimes\lambda_\alpha$ coincides with the outer measure induced by m.*

Proof. The first half of Lemma 3 is well-known (see for example [1, §13 and (15.17. *j*)]), hence it suffices to prove the second half.

Let E be any m-measurable subset of H. Then it is known that there exist Baire subsets E_1 and E_2 of H such that $E_1 \subset E \subset E_2$ and $m(E_2 - E_1) = 0$ (see [1, (19.30)]). Here we call B a Baire subset of H if

B is a member of the σ-field generateded by the open subsets of H written in the form $\{x \in H \mid f(x) > 0\}$ by some real-valued continuous function f on H. Let V be an open subset of H written in the above form. Then V is σ-closed. Since H is compact, V is σ-compact. Then it is easy to see that V is a countable union of open sets which are members of $\otimes\mathfrak{M}_\alpha$. This implies that every Baire subset of H belongs to $\otimes\mathfrak{M}_\alpha$. This together with the first half of Lemma 3 implies that

$$\otimes\lambda_\alpha(E_1) = m(E) = \otimes\lambda_\alpha(E_2).$$

The second half of Lemma 3 is now obvious.

Theorem 3. *There exist a compact non-metrizable abelian group H with the normalized Haar measure m and a sequence $\{T_n\}$ of invertible m-measure-preserving transformations in H such that $\{T_n\}$ converges to the identity transformation I in H, but for any subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ $\lim_{n\to\infty} T_{k(n)}x$ does not exist for any x in H.*

Proof. Let K be the circle group and $(K, \mathfrak{M}, \lambda)$ the normalized Lebesgue measure space. We define a sequence $\{S_n\}$ of invertible λ-measure-preserving transformations in K as follows:

$$S_1 \exp(it) = \begin{cases} \exp(i(t+\pi)) & \text{if } 0 \leqq t < \pi/2 \text{ or } \pi \leqq t < \pi + \pi/2 \\ \exp(it) & \text{if } \pi/2 \leqq t < \pi \text{ or } \pi + \pi/2 \leqq t < 2\pi, \end{cases}$$

$$S_2 \exp(it) = \begin{cases} \exp(it) & \text{if } 0 \leqq t < \pi/2 \text{ or } \pi \leqq t < \pi + \pi/2 \\ \exp(i(t+\pi)) & \text{if } \pi/2 \leqq t < \pi \text{ or } \pi + \pi/2 \leqq t < 2\pi, \end{cases}$$

$$S_3 \exp(it) = \begin{cases} \exp(i(t+\pi)) & \text{if } 0 \leqq t < \pi/4 \text{ or } \pi \leqq t < \pi + \pi/4 \\ \exp(it) & \text{if } \pi/4 \leqq t < \pi \text{ or } \pi + \pi/4 \leqq t < 2\pi, \end{cases}$$

$$S_4 \exp(it) = \begin{cases} \exp(it) & \text{if } 0 \leqq t < \pi/4,\ \pi/2 \leqq t < \pi + \pi/4 \\ & \text{or } \pi + \pi/2 \leqq t < 2\pi \\ \exp(i(t+\pi)) & \text{if } \pi/4 \leqq t < \pi/2 \text{ or } \pi + \pi/4 \leqq t < \pi + \pi/2, \end{cases}$$

and so on.

It is obvious that $\{S_n\}$ converges to the identity transformation in K in measure, but $\lim_{n\to\infty} S_n x$ does not exist for any x in K. Let $\mathfrak{S}$ be the set of all subsequences $\{k(n)\}$ of $\{n\}$. We note that the cardinal number of $\mathfrak{S}$ is equal to $2^{\aleph_0}$. We consider the product measure space $(\otimes K_{\{k(n)\}}, \otimes\mathfrak{M}_{\{k(n)\}}, \otimes\lambda_{\{k(n)\}})$ of $(K_{\{k(n)\}}, \mathfrak{M}_{\{k(n)\}}, \lambda_{\{k(n)\}})$, where $(K_{\{k(n)\}}, \mathfrak{M}_{\{k(n)\}}, \lambda_{\{k(n)\}}) = (K, \mathfrak{M}, \lambda)$ for all $\{k(n)\} \in \mathfrak{S}$. Then the compact abelian group $H \equiv \otimes K_{\{k(n)\}}$ is not metrizable. In fact there is no countable open basis at the identity of H, and so H is not metrizable.

For each $\{k(n)\} \in \mathfrak{S}$ we define a sequence $\{S_j^{\{k(n)\}}\}$ of invertible λ-measure-preserving transformations as follows: $S_j^{\{k(n)\}} = S_1$ if $j \leqq k(1)$, and $S_j^{\{k(n)\}} = S_m$ if $k(m-1) < j \leqq k(m)$. For each $j (j = 1, 2, 3, \cdots)$, let T_j be a transformation of H onto H defined by

$$T_j x = (S_j^{\{k(n)\}} x_{\{k(n)\}})_{\{k(n)\}} \tag{9}$$

for $x = (x_{\{k(n)\}})_{\{k(n)\}} \in \mathfrak{S}$. Then $\{T_j\}$ is a sequence of invertible $\otimes\lambda_{\{k(n)\}}$-

measure-preserving transformations in H.

On the other hand, by Lemma 3 $\otimes\lambda_{\{k(n)\}}$ is the restriction of the normalized Haar measure m H to the σ-field $\otimes\mathfrak{M}_{\{k(n)\}}$ and the outer measure induced by $\otimes\lambda_{\{k(n)\}}$ coincides with the outer measure induced by m. Thus $\{T_j\}$ is a sequence of invertible m-measure-preserving transformations in H. Let V be a neighborhood of the identity of H in the form $\otimes V_{\{k(n)\}}$, where $V_{\{k(n)\}}$ is an open neighborhood of the identity of $K_{\{k(n)\}}$, but it coincides with $K_{\{k(n)\}}$ except for finitely many coordinates $\{k(n)\}\in\mathfrak{S}$. Let I be the identity transformation in H. Since $\{S_j\}$ converges to the identity transformation in K, it is easily seen that

$$\lim_{j\to\infty} m\{x\in H\,|\,(T_jx)(Ix)^{-1}\notin V\}=0. \tag{10}$$

This implies that $\{T_j\}$ converges to I in measure (in reference to the definition of convergence in measure in general case, see [2, Definition 1]). By virtue of [2, Theorem 1], $\{T_j\}$ converges to I weakly. But from the construction of $\{T_j\}$, for any subsequence $\{T_{k(j)}\}$ of $\{T_j\}$ $\lim_{j\to\infty} T_{k(j)}x$ does not exist for any x in H. The proof is completed.

Acknowledgment. I would like to thank Professor Shigeru Tsurumi who read an earlier version of this paper and made some helpful suggestions.

References

[1] E. Hewitt and K. A. Ross: Abstract Harmonic Analysis, Vol. 1. Berlin (1963).

[2] F. Papangelou: On weak convergence and convergence in measure of transformations in topological measure spaces. J. London Math. Soc., **43**, 521–526 (1968).