8. A Note on Filipov's Theorem

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V. V. Filipov [1] proved the following theorem:

Filipov's Theorem. *A paracompact M-space with a point-countable open basis is metrizable.*[2)]

On the other hand A. Okuyama [4] proved

Okuyama's Theorem. *A space X is metrizable if and only if it is a paracompact M-space, and the diagonal of the product space $X \times X$ is a G_δ-set.*

These two metrization theorems for an M-space look like to be considerably different, but the fact is that we can easily form a theorem which includes both of them as corollaries.

Theorem. *A space X is metrizable if and only if it is a paracompact M-space with a point-countable collection $\mathcal{U}$ of open sets such that for any different points x and y of X there is $U \in \mathcal{U}$ satisfying $x \in U$ and $y \notin U$.*

Proof. We shall prove only the sufficiency. The proof is a slight modification of Filipov's, and we make a full use of the following Miščenko's theorem [2] as Filipov did:

Miščenko's Theorem. *Let $\mathcal{U}$ be a point-countable collection of subsets of a set X and X' a subset of X. Then there are at most countably many finite minimal covers (=coverings) of X' by elements of $\mathcal{U}$, where we mean by a minimal cover a cover which contains no proper subcover.*

Now let us assume that X is a space satisfying the conditions in the theorem. Since X is a paracompact M-space, there is a metric space Y and a perfect mapping f from X onto Y. Note that for each $x \in X f^{-1}f(x)$ is a compact set of X, and we shall denote this set by F_x throughout this paper. For each natural number n we denote by $\mathcal{V}_n$ a locally finite open cover of Y such that *mesh* $\mathcal{V}_n = sup$ {*diameter of*

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2) Actually he used the terminology '*p-space*' instead of '*M-space*', but for a paracompact space the two concepts, M-space (due to K. Morita) and p-space (due to A. Arhangelskii) coincide with each other, and a paracompact M-space is characterized as the inverse image of a metric space by a perfect mapping. As for terminologies and symbols in this paper see J. Nagata [3]. Also note that all spaces in this paper are Hausdorff spaces.

$V \mid V \in \mathcal{V}_n\} < 1/n$.

First we shall prove that X is perfectly normal. Suppose G is a given closed set of X. Let V be an element of $\mathcal{V}_n$ for which $f^{-1}(V) \cap G \neq \phi$. Then by Miščenko's theorem, there are at most countably many finite minimal covers $\mathcal{U}_m(V)$, $m=1, 2, \cdots$ of $f^{-1}(V) \cap G$ by members of $\mathcal{U}$. (If there are only finitely many of them, then just repeatedly count a cover.) Put

$$U_m(V) = \cup\{U \cap f^{-1}(V) \mid U \in \mathcal{U}_m(V)\},\ U'_m(V) = U_1(V) \cap \cdots \cap U_m(V),$$
$$U_{m,n} = \cup\{U'_m(V) \mid V \in \mathcal{V}_n,\ f^{-1}(V) \cap G \neq \phi\}.$$

Then $U_{m,n}$, $m, n=1, 2, \cdots$ are open sets satisfying $U_{m,n} \supset G$ (assume that $X \in U$). We can show that $G = \bigcap_{m,n=1}^{\infty} U_{m,n}$. To do so let $y \notin G$. If $F_y \cap G = \phi$, then $f(y) \notin f(G)$, and hence there is n for which $S(f(y), \mathcal{V}_n) \cap f(G) = \phi$ in Y. This implies that $S(G, f^{-1}(\mathcal{V}_n)) \cap F_y = \phi$ in X. Therefore $U_{m,n} \cap F_y = \phi$ for any m since $U_{m,n} \subset S(G, f^{-1}(\mathcal{V}_n))$ follows from the definition of $U_m(V)$. Thus $y \notin U_{m,n}$. On the other hand, if $F_y \cap G \neq \phi$, then to each point x of $F_y \cap G$ we assign a member U of $\mathcal{U}$ such that $x \in U$ and $y \notin U$. Since $F_y \cap G$ is compact, we can cover it by finitely many U's, say $U_1, \cdots, U_k$. We may assume that $\mathcal{U}' = \{U_1, \cdots, U_k\}$ is a minimal cover of $F_y \cap G$. Then F_y is contained in the open set $\bigcup_{i=1}^{k} U_i \cup (X - G)$. Thus for some n $S(F_y, f^{-1}(\mathcal{V}_n)) \subset \bigcup_{i=1}^{k} U_i \cup (X - G)$. Since $\mathcal{V}_n$ is point-finite, $\{V \mid V \in \mathcal{V}_n,\ f^{-1}(V) \cap F_y \neq \phi,\ f^{-1}(V) \cap G \neq \phi\}$ is finite; we denote this open collection by $\{V_1, \cdots, V_l\}$. Note that $F_y \cap G \subset f^{-1}(V_i) \cap G \subset \bigcup_{i=1}^{k} U_i$, and hence $\mathcal{U}'$ is a finite minimal cover of $f^{-1}(V_i) \cap G$ for each i, and hence $\mathcal{U}' = \mathcal{U}_{m_i}(V_i)$, $i=1, \cdots, l$, for some m_i. Put $m = \max\{m_1, \cdots, m_l\}$. To prove $y \notin U_{m,n}$, recall that $U_{m,n}$ is the union of $U'_m(V)$ for all $V \in \mathcal{V}_n$ satisfying $f^{-1}(V) \cap G \neq \phi$. If $f^{-1}(V) \cap F_y = \phi$, then $U'_m(V) \cap F_y = \phi$, because $U'_m(V) \subset f^{-1}(V)$. Hence $y \notin U'_m(V)$. If $f^{-1}(V) \cap F_y \neq \phi$, then $V = V_i$ for some i, and hence

$$U'_m(V) \subset U'_{m_i}(V_i) \subset U_{m_i}(V_i) = \cup\{U_j \cap f^{-1}(V_i) \mid j=1, \cdots, k\} \not\ni y.$$

Thus $y \notin U'_m(V)$ is concluded in either case proving that $y \notin U_{m,n}$. This completes the proof that $G = \bigcap_{m,n=1}^{\infty} U_{m,n}$, and hence X is perfectly normal.

Now, let us turn to the proof that X is metrizable. To begin with we shoud note that we may assume that for any different points x and y of X there is $U \in \mathcal{U}$ satisfying $x \in U \subset \bar{U} \not\ni y$. Because by virtue of the perfect normality of X we can express each $U \in \mathcal{U}$ as $U = \bigcup_{n=1}^{\infty} \bar{U}_n = \bigcup_{n=1}^{\infty} U_n$ with open sets U_n, $n=1, 2, \cdots$ to add those U_n to $\mathcal{U}$ without changing its point-countability. Furthermore we may assume that the intersection of any finitely many members of $\mathcal{U}$ belongs to $\mathcal{U}$.

Let $V \in \mathcal{V}_n$. Then again by use of Miščenko's theorem there are at most countably many finite minimal covers of $f^{-1}(V)$ by members of $\mathcal{U}$. We denote all those covers by $\mathcal{W}_m(V)$, $m=1, 2, \cdots$ and their restrictions to $f^{-1}(V)$ by $\mathcal{W}'_n(V)$ to put $\mathcal{W}_{m,n} = \cup\{\mathcal{W}'_m(V) \mid V \in \mathcal{V}_n\}$. Then $\mathcal{W}_{m,n}$ is a locally finite open collection in X since $\mathcal{V}_n$ is locally finite in Y. Let us prove that $\bigcup_{m,n=1}^{\infty} \mathcal{W}_{m,n}$ is an open basis of X. Assume N is an open nbd ($=$neighborhood) of a point x of X. Then $F_x - N$ is a compact set which does not contain x. Thus there are $U'_1, \cdots, U'_k \in \mathcal{U}$ such that $x \in U'_1 \cap \cdots \cap U'_k \subset \bar{U}'_1 \cap \cdots \cap \bar{U}'_k \subset X - (F_x - N)$. Hence $U_0 = U'_1 \cap \cdots \cap U'_k$ is a member of $\mathcal{U}$ satisfying $x \in U_0 \subset \bar{U}_0 \subset X - (F_x - N)$. Cover the compact set $F_x - U_0$ with $U_1, \cdots, U_l \in \mathcal{U}$ such that $x \notin \bigcup_{i=1}^{l} U_i$. We may assume $\{U_0, U_1, \cdots, U_l\}$ is a minimal cover of F_x. Now $W = U_0 \cup U_1 \cup \cdots \cup U_l - (\bar{U}_0 - N)$ is an open nbd of F_x, because $(\bar{U}_0 - N) \cap F_x = \phi$ follows from $\bar{U}_0 \subset X - (F_x - N)$. Thus there is n and $V \in \mathcal{V}_n$ for which $F_x \subset f^{-1}(V) \subset W$. Since $\{U_0, \cdots, U_l\}$ is a finite minimal cover of $f^{-1}(V)$, it should be denoted by $\mathcal{W}_m(V)$ for some m. Hence $f^{-1}(V) \cap U_0 \in \mathcal{W}'_m(V) \subset \mathcal{W}_{m,n}$. It is obvious that $x \in f^{-1}(V) \cap U_0$, while $f^{-1}(V) \cap U_0 \subset N$ follows from $f^{-1}(V) \subset W$ and $W \cap U_0 \subset W \cap \bar{U}_0 \subset N$. Therefore $\bigcup_{m,n=1}^{\infty} \mathcal{W}_{m,n}$ is a σ-locally finite open basis of X which assures the metrizability of X by Nagata-Smirnov's theorem.

Remark. This theorem obviously includes Filipov's theorem as a corollary. On the other hand it is a well-known fact that a paracompact space X with a G_δ-diagonal in $X \times X$ has a sequence $\mathcal{U}_1, \mathcal{U}_2, \cdots$ of locally finite open covers such that for any different points x and y of X there is n for which $y \notin S(x, \mathcal{U}_n)$. Thus Okuyama's theorem also turns out to be a corollary of ours.

The condition for $\mathcal{U}$ in our theorem is a considerably tolerant one among the conditions to be satisfied by an open collection, and this fact leads us to think of applying the theorem on other metrization problems. For example, we obtain

Corollary. *Let X be a paracompact, locally M-space. If X has a cover $\{F_\alpha \mid \alpha \in A\}$ of closed, G_δ, metrizable sets F_α, $\alpha \in A$ such that for some open sets U_α containing F_α, $\{U_\alpha \mid \alpha \in A\}$ is point-countable, then X is metrizable.*

Proof. It suffices to prove the assertion in the special case that the whole space X is M. The proof for the special case is quite easy and left to the reader.

Remark. This corollary does not look so beautiful, but it contains the following theorems due to A. Okuyama [5] and A. H. Stone (see J. Nagata [3]), respectively.

Let X be a collectionwise normal, locally M-space. If X is the union of countably many closed metrizable sets, then it is metrizable.

Let X be a collectionwise normal, locally countably compact space. If X is the union of countably many closed metrizable sets, then it is metrizable.

The former theorem is a generalization of the latter because every countably compact space is M. It is easily seen that a space satisfying the conditions in the former theorem is paracompact and perfectly normal, and hence our corollary slightly generalizes the theorem.

References

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