30. A Note on Radicals of Ideals in Nonassociative Rings

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Let R be a nonassociative ring and let $\mathfrak{A}=\{u=\mathfrak{P}_n^{(\nu)}\}$ be the set of all formal nonassociative products.[1] In [3], Brown-McCoy has defined that an ideal[2] P is a u-prime ideal, if whenever $u(A_1, \cdots, A_n)$ is contained in P for ideals A_i of R, then at least one of the ideals A_i is contained in P. We shall generalize this concept as follows: Let $\mathfrak{U}$ be any fixed subset of $\mathfrak{A}$. *An ideal P is said to be $\mathfrak{U}$-ideal if whenever $\Sigma_{\mathfrak{P}_n^{(\nu)} \in \mathfrak{U}} \mathfrak{P}_n^{(\nu)}(A_{\nu 1}, \cdots, A_{\nu n})$ is contained in P, where Σ denotes the restricted sum and $A_{\nu i}$ are ideals, then $A_{\nu i}$ is contained in P for some ν, i.* It is the aim of this paper to investigate $\mathfrak{U}$-ideals and to present some related results.

In section 1, $\mathfrak{U}$-systems are defined by analogy with m-systems introduced in [4]. If A is an ideal of R, a $\mathfrak{U}$-radical $\mathfrak{U}(A)$ of the ideal A is defined to be the set of all elements r of R with the property that every $\mathfrak{U}$-system which contains an element of A. We shall prove that $\mathfrak{U}(A)$ is the intersection of all $\mathfrak{U}$-ideals which contains A. Section 2 lays definitions of $\mathfrak{U}^*$-ideals and $\mathfrak{U}^*$-radicals of ideals which are analogous to those of u^*-prime ideals and u^*-radicals of ideals in [3]. We shall show that always $\mathfrak{U}(A)=\mathfrak{U}^*(A)$ under the assumption that $\mathfrak{U}$ is a finite subset of $\mathfrak{A}$, where $\mathfrak{U}^*(A)$ is the $\mathfrak{U}^*$-radical of an ideal A. In the fininal section we define a $\mathfrak{U}$-radical of the ring R, which is denoted by $\mathfrak{U}(R)$, as the one of the zero ideal of R, and show that $\mathfrak{U}(R)$ has the usual properties expected of a radical. Moreover we shall show that $\mathfrak{U}(R_n)=(\mathfrak{U}(R))_n$, where R_n and $(\mathfrak{U}(R))_n$ are the total matric rings of order n with coefficients in R and $\mathfrak{U}(R)$ respectively.

1. $\mathfrak{U}$-ideals and $\mathfrak{U}$-radicals.

Throughout this paper, we let $\mathfrak{U}$ be any fixed subset of $\mathfrak{A}$. The principal ideal generated by an element a of R will be denoted by (a). The complement of an ideal in R will be denoted by $C(A)$.

Lemma 1. *Let P be an ideal of R. Then the following three conditions are equivalent:*

(i) *P is a $\mathfrak{U}$-ideal.*

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1) Following Behrens [1, 2], we shall denote by $\mathfrak{P}_n^{(\nu)}(A_1, \cdots, A_n)$ a fixed type ν of the product of ideals $A_1, \cdots, A_n$ in R.

2) The word "ideal" will always mean a "two-sided ideal."

(ii) $\Sigma\mathfrak{P}_n^{(\nu)}(A_{\nu 1}, \cdots, A_{\nu n}) \cap C(P)$ *is non-void,*[3] *if* $C(P) \cap A_{\nu i}$ *are non-void for all* ν *and* i, *where* $A_{\nu i}$ *are ideals of* R.

(iii) $\Sigma\mathfrak{P}_n^{(\nu)}((a_{\nu 1}), \cdots, (a_{\nu n})) \cap C(P)$ *is non-void, if* $a_{\nu i} \in C(P)$ *for all* ν *and* i.

Definition 1. A subset M of R is a $\mathfrak{U}$-*system*, if $A_{\nu i}$ are ideals of R, each of which meets M, then $\Sigma\mathfrak{P}_n^{(\nu)}(A_{\nu 1}, \cdots, A_{\nu n})$ meets M.

Clearly an ideal P of R is a $\mathfrak{U}$-ideal if and only if $C(P)$ is a $\mathfrak{U}$-system. Suppose that $u \in \mathfrak{U}$, then a u-prime ideal in the sense of [3] is a $\mathfrak{U}$-ideal. But the converse need not be true. For, let R be the algebra in the example 1 of [3]. If we put $\mathfrak{U}=\{u, u'\}$, where $u(x_1, x_2, x_3) =(x_1x_2)x_3$ and $u'(x_1, x_2, x_3)=x_1(x_2x_3)$, then (0) is u'-prime. Hence (0) is $\mathfrak{U}$-prime. However (0) is not u-prime.

Theorem 1. *Let* M *be a* $\mathfrak{U}$-*system in* R, *and* A *an ideal which does not meet* M. *Then* A *is contained in an ideal* P *which is maximal in the class of ideals which do not meet* M. *The ideal* P *is necessarily a* $\mathfrak{U}$-*ideal.*

Proof. The existence of P follows at once from Zorn's lemma. We now show that P is a $\mathfrak{U}$-ideal. Suppose that $A_{\nu i}$ is not contained in P, then the maximal property of P implies that each of the ideals $P+A_{\nu i}$ meets M. By Definition 1 it follows that $\Sigma\mathfrak{P}_n^{(\nu)}(A_{\nu 1}+P, \cdots, A_{\nu n}+P)$ meets M. But clearly we have that $\Sigma\mathfrak{P}_n^{(\nu)}(A_{\nu 1}+P, \cdots, A_{\nu n}+P) \subseteq P+\Sigma\mathfrak{P}_n^{(\nu)}(A_{\nu 1}, \cdots, A_{\nu n})$. Since P does not meet M, $\Sigma\mathfrak{P}_n^{(\nu)}(A_{\nu 1}, \cdots, A_{\nu n})$ is not contained in P, and hence $\Sigma\mathfrak{P}_n^{(\nu)}(A_{\nu 1}, \cdots, A_{\nu n})$ meets $C(P)$. By Lemma 1, this shows that P is a $\mathfrak{U}$-ideal.

Definition 2. If A is an ideal of R, the $\mathfrak{U}$-*radical* $\mathfrak{U}(A)$ of A is the set of all elements r of R such that every $\mathfrak{U}$-system which contains r meets A.

Theorem 2. *If* A *is an ideal of* R, $\mathfrak{U}(A)$ *is the intersection of all* $\mathfrak{U}$-*ideals, each of which contains* A.

Proof. Clearly A is contained in $\mathfrak{U}(A)$. Furthermore, A and $\mathfrak{U}(A)$ are contained in the same $\mathfrak{U}$-ideals. For, suppose that $A_{\nu i}$ is contained in P, where P is a $\mathfrak{U}$-ideals, and that $r \in \mathfrak{U}(A)$. If r is not in P, then $C(P)$ must contain an element of A, since $C(P)$ is a $\mathfrak{U}$-system. But clearly $C(P) \cap A$ is void. Thus $r \in P$ and hence $\mathfrak{U}(A)$ is contained in P as desired. This shows that $\mathfrak{U}(A)$ is contained in the intersection of all the $\mathfrak{U}$-ideals, each of which contains A. To prove the converse inclusion, let a be an element of R, but not in $\mathfrak{U}(A)$. Then by the definition of $\mathfrak{U}(A)$, there exists a $\mathfrak{U}$-system M which contains a but does not meet A. By Theorem 1, there exists a $\mathfrak{U}$-ideal containing A which does not meet M, and therefore does not contain a. Hence a can not be in the intersection of all $\mathfrak{U}$-ideals containing A.

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3) The symbol "Σ" will always mean the restricted sum $\Sigma_{\mathfrak{P}_n^{(\nu)} \in \mathfrak{U}}$.

Corollary. *The $\mathfrak{U}$-radical of an ideal is an ideal.*

2. Equivalence of $\mathfrak{U}$-radical and $\mathfrak{U}^*$-radical.

Throughout this section, let $\mathfrak{U}$ be a finite subset of $\mathfrak{A}:\mathfrak{U}=\{\mathfrak{P}_{n(1)}^{(\nu(1))}, \cdots, \mathfrak{P}_{n(m)}^{(\nu(m))}\}$ and put $k=n(1)+\cdots+n(m)$. We set $\mathfrak{U}^*((x))=\Sigma_{i=1}^m \mathfrak{P}_{n(i)}^{(\nu(i))}((x), \cdots, (x))$, for all $x \in R$. Then, we can define a $\mathfrak{U}^*$-ideal and the $\mathfrak{U}^*$-radical $\mathfrak{U}^*(A)$ of an ideal A, which are analogous to the $\mathfrak{U}$-ideal and the $\mathfrak{U}$-radical of A, respectively.

Lemma 2. *If a is an element of a $\mathfrak{U}^*$-system M^*, then there exists a $\mathfrak{U}$-system M such that $a \in M \subseteq M^*$.*

Proof. Let $M=\{a_1, a_2, \cdots\}$, where $a_1=a$ and the other elements of M are defined inductively as follows. Since $a_1 \in M^*$, $\mathfrak{U}^*((a_1))$ meets M^*. Let $a_2 \in \mathfrak{U}^*((a_1)) \cap M^*$. Then let, in general, $a_k \in \mathfrak{U}^*((a_{k-1})) \cap M^*$. We can prove that M is a $\mathfrak{U}$-system. Let $a_{i(1)}, \cdots, a_{i(k)} \in M$ and assume that $i(k)$ is the maximal number in $\{i(1), \cdots, i(k)\}$. Then we have $a_{i(k+1)} \in \mathfrak{U}^*((a_{i(k)}))=\Sigma_{i=1}^m \mathfrak{P}_{n(i)}^{(\nu(i))}((a_{i(k)}), \cdots, (a_{i(k)})) \subseteq \mathfrak{P}_{n(1)}^{(\nu(1))}((a_{i(1)}), \cdots, (a_{i(n(1))}))+\cdots+\mathfrak{P}_{n(m)}^{(\nu(m))}((a_{i(n(1))}+\cdots+n(m-1))+1), \cdots, (a_{i(k)}))$. Hence M is a $\mathfrak{U}$-system containing a.

Theorem 3. *If A is an ideal of R, then $\mathfrak{U}(A)=\mathfrak{U}^*(A)$.*

Proof. Clearly a $\mathfrak{U}$-ideal is a $\mathfrak{U}^*$-ideal, and hence we have $\mathfrak{U}^*(A) \subseteq \mathfrak{U}(A)$. The converse inclusion is immediate by Lemma 2.

Corollary. *If A is an ideal of R, then $A=\mathfrak{U}^*(A)$ if and only if A is an intersection of $\mathfrak{U}$-ideals.*

3. The $\mathfrak{U}$-radical of a ring.

Definition 3. The *$\mathfrak{U}$-radical* of a ring R is the $\mathfrak{U}$-radical of the zero ideal in the ring R. In symbol: $\mathfrak{U}(R)$.

Definition 4. An element a of a ring is *nilpotent* if $u(a, \cdots, a)=0$ for some $u \in \mathfrak{A}$. An ideal is a *nil ideal* if each of its element is nilpotent.

For each $u \in \mathfrak{U}$, an u-prime ideal is a $\mathfrak{U}$-ideal. Hence the u-radical N_u of the ring R in the sense of [3] contains $\mathfrak{U}(R)$. By §5 in [3], N_u is a nil ideal of R. Hence the $\mathfrak{U}$-radical $\mathfrak{U}(R)$ of the ring R is also a nil ideal of R.

Definition 5. A ring R is said to be a *$\mathfrak{U}$-ring* if and only if (0) is a $\mathfrak{U}$-ideal of R.

If P is a $\mathfrak{U}$-ideal, then R/P is a $\mathfrak{U}$-ring and conversely. Since $\mathfrak{U}(R)$ is the intersection of all the $\mathfrak{U}$-ideals of R, by the similar methods as in Theorems 5 and 6 of [4], we have the following two theorems.

Theorem 4. *If $\mathfrak{U}(r)$ is the $\mathfrak{U}$-radical of R, $R/\mathfrak{U}(R)$ has the zero $\mathfrak{U}$-radical.*

Theorem 5. *A necessary and sufficient condition that a ring be isomorphic to a subdirect sum of $\mathfrak{U}$-rings is that it has zero $\mathfrak{U}$-radical.*

Lemma 3. *Let S be an over ring of a ring R. If each ideal of*

R is also an ideal of S, then $\mathfrak{U}(R)=\mathfrak{U}(S)\cap R$.

Proof. It is easily shown that if P is a $\mathfrak{U}$-ideal of S, then $P\,R$ is a $\mathfrak{U}$-ideal of R. Hence we have that $\mathfrak{U}(R)\subseteq\mathfrak{U}(S)\cap R$ by Theorem 2. The converse inclusion is immediate, because a $\mathfrak{U}$-system in R is a $\mathfrak{U}$-system in S.

If R is a ring, we shall denote by R_n the ring of all matrices of order n with coordinates in R.

Theorem 6. *Let R be a ring with unit element. Then R is a $\mathfrak{U}$-ring if and only if R_n is a $\mathfrak{U}$-ring.*

Proof. First we assume that R is not a $\mathfrak{U}$-ring. Suppose that $\Sigma\mathfrak{P}_n^{(\nu)}((a_1^{(\nu)}),\cdots,(a_n^{(\nu)}))=0$, where each $a_i^{(\nu)}$ is a nonzero element of R, then $\mathfrak{P}_n^{(\nu)}((a_1^{(\nu)}),\cdots,(a_n^{(\nu)}))=0$ for each $\mathfrak{P}_n^{(\nu)}\in\mathfrak{U}$. If e_{ik} is the matrix units in R_n, then by Lemma of [3], it follows that $\mathfrak{P}_n^{(\nu)}((a_1^{(\nu)}e_{11}),\cdots,(a_n^{(\nu)}e_{11}))=0$, where each $a_i^{(\nu)}e_{11}\neq 0$ and therefore $\Sigma\mathfrak{P}_n^{(\nu)}((a_1^{(\nu)}e_{11}),\cdots,(a_n^{(\nu)}e_{11}))=0$. Thus we see that R is not a $\mathfrak{U}$-ring. Conversely, suppose that R_n is not a $\mathfrak{U}$-ring and that $\Sigma\mathfrak{P}_n^{(\nu)}(a_1^{(\nu)},\cdots,A_n^{(\nu)})=0$, where each $A_i^{(\nu)}$ is not a nonzero ideal of R_n, then $\mathfrak{P}_n^{(\nu)}(A_1^{(\nu)},\cdots,A_n^{(\nu)})=0$ for each $\mathfrak{P}_n^{(\nu)}\in\mathfrak{U}$. By Lemma of [3], there exist nonzero elements $a_1^{(\nu)},\cdots,a_n^{(\nu)}$ in R such that $\mathfrak{P}_n^{(\nu)}((a_1^{(\nu)}),\cdots,(a_n^{(\nu)}))=0$ for each $\mathfrak{P}_n^{(\nu)}\in\mathfrak{U}$. Hence $\Sigma\mathfrak{P}_n^{(\nu)}((a_1^{(\nu)}),\cdots,(a_n^{(\nu)}))=0$. This shows that R is not a $\mathfrak{U}$-ring.

Theorem 7. *If R is any nonassociative ring, then* $\mathfrak{U}(R_n)=(\mathfrak{U}(R))_n$.

Proof. This is immediate by Lemma 3 and Theorem 6.

References

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