52. *Realization of Irreducible Bounded Symmetric Domain of Type (V)*

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This note is a partial report of our researches concerning irreducible bounded symmetric domains of exceptional type. The detailed exposition with full proofs will be presented elsewhere.

1. It is well-known, since E. Cartan [1], that the irreducible bounded symmetric domain of type (V) is of 16-dimension and that of type (VI) is of 27-dimension; however no explicit descriptions of the *bounded* models of these domains have been clarified, as far as we know. In fact, several authors, M. Koecher, Pyateskii-Shapiro, U. Hirzebruch have tried to give the *unbounded* models of the domains using the theory of Jordan algebras and homogeneous cones. But, to determine the corresponding bounded models from the unbounded ones seems to be not so easy, though the general concept of Cayley transforms of bounded symmetric domains has been established by A. Koranyi and J. A. Wolf [6].

On the other hand, we have developed in [5] the *canonical* method of bounded realizations of general bounded symmetric domains as matrix-spaces; this yields, as special cases, the well-known bounded models of irreducible bounded symmetric domains of classical type (I)-(IV). We will apply it to the domain of type (V), and give the bounded model which is the *simplest one* in our sense.

2. Now we settle the notation that will be used. Let $\mathfrak{C}$ denote the algebra of Cayley numbers over the complex numbers C and $\mathfrak{J}$ the corresponding simple Jordan algebra of exceptional type [2], [8]; namely the complex vector space of all hermitian matrices of degree 3 over $\mathfrak{C}$ whose elements are written as follows:

$$(1)\qquad u=\begin{pmatrix}\xi_1, x_3, \bar{x}_2\\ \bar{x}_3, \xi_2, x_1\\ x_2, \bar{x}_1, \xi_3\end{pmatrix};\ x_i\in\mathfrak{C},\ \xi_i\in C.$$

($x\to\bar{x}$ denotes the involution in the sense of Cayley numbers). The Jordan product $u\cdot v$ in $\mathfrak{J}$ is, as usual, given by $u\cdot v=\frac{1}{2}(uv+vu)$, where uv is the ordinary matrix product. C. Chevalley and R. D. Schafer proved in [2] that the 27-dimensional irreducible representation $\mathfrak{G}$ of the complex simple Lie algebra $\mathfrak{g}_C$ of type E_6 are realized over the representation space $\mathfrak{J}$ as follows:

$$\mathfrak{G}=\mathfrak{D}\oplus\mathfrak{R} \quad (\oplus \text{ denotes the direct sum}),$$

where $\mathfrak{D}$ denotes the derivation algebra of $\mathfrak{J}$ and $\mathfrak{R}$ the set of all (right) translations by elements with zero traces. Our first task is to establish the so-called *symmetric pair* corresponding to the irreducible symmetric space of type E III (=the irreducible bounded symmetric domain of type (V)) in the classification table of E. Cartan (see [4], Chap. IX). For this sake, we now introduce some notation. $\bar{\mathfrak{J}}$ being the space of all skew-hermitian matrices u' of degree 3 over $\mathfrak{C}$;

$$u'=\begin{pmatrix} z_1, & x_3, & -\bar{x}_2 \\ -\bar{x}_3, & z_2, & x_1 \\ x_2, & -\bar{x}_1, & z_3 \end{pmatrix} \tag{2}$$

($x_i \in \mathfrak{C}$, $z_i \in \mathfrak{C}$: $\bar{z}_i=-z_i$), D_u ($u \in \bar{\mathfrak{J}}$) denotes the derivation of $\mathfrak{J}$: $v \to \frac{1}{2}[u, v]=\frac{1}{2}(uv-vu)$, and R_u ($u \in \mathfrak{J}$) the translation of $\mathfrak{J}$: $v \to v\cdot u =\frac{1}{2}(uv+vu)$. Now, we write the element u in (1) as

$$u=\xi_1 e_1+\xi_2 e_2+\xi_3 e_3+x_1 u_1+x_2 u_2+x_3 u_3$$

($\xi_i \in C$, $x_i \in \mathfrak{C}$), and the element u' in (2) as

$$u'=z_1 e_1+z_2 e_2+z_3 e_3+x_1\bar{u}_1+x_2\bar{u}_2+x_3\bar{u}_3$$

Then we can decompose $\mathfrak{D}$ and $\mathfrak{R}$ in the following way;

$$\mathfrak{D}=\mathfrak{D}_0\oplus\mathfrak{D}_1\oplus\mathfrak{D}_2\oplus\mathfrak{D}_3,$$
$$\mathfrak{R}=\mathfrak{R}_0\oplus\mathfrak{R}_1\oplus\mathfrak{R}_2\oplus\mathfrak{R}_3,$$

where, $\mathfrak{D}_0$ is the subalgebra generated by $\{\sum_{i=1}^3 D_{z_i e_i};\ z_i \in \mathfrak{C},\ \bar{z}_i=-z_i,\ \sum_{i=1}^3 z_i=0\}$, $\mathfrak{D}_i=\{D_{x\bar{u}_i};\ x_i \in \mathfrak{C}\ (1\leq i\leq 3)\}$, $\mathfrak{R}_0=\{\sum_{i=1}^3 R_{\xi_i e_i};\ \sum \xi_i=0,\ \xi_i \in C\}$ and $\mathfrak{R}_i=\{R_{xu_i};\ x \in \mathfrak{C}\ (1\leq i\leq 3)\}$. Then it is readily seen that $\dim_c \mathfrak{D}_i =\dim_c \mathfrak{R}_i=8$ $(1\leq i\leq 3)$, $\dim_c \mathfrak{D}_0=28$, $\dim_c \mathfrak{R}_0=2$ (hence $\dim_c \mathfrak{D}=52$, $\dim_c \mathfrak{R}=26$).

3. Under the above notation, we have a symmetric pair of type E III: $\mathfrak{G}=\mathfrak{K}\oplus\mathfrak{M}$ ($[\mathfrak{K},\mathfrak{K}]\cong\mathfrak{o}(10, C)$) in the following way:

Proposition 1. $\mathfrak{K}=\mathfrak{D}_0\oplus\mathfrak{D}_1\oplus\mathfrak{R}_0\oplus\mathfrak{R}_1$, $\mathfrak{M}=\mathfrak{D}_2\oplus\mathfrak{D}_3\oplus\mathfrak{R}_2\oplus\mathfrak{R}_3$ ($[\mathfrak{K},\mathfrak{K}] \subset\mathfrak{K}$, $[\mathfrak{K},\mathfrak{M}]\subset\mathfrak{M}$, $[\mathfrak{M},\mathfrak{M}]\subset\mathfrak{K}$).

Now let $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{K}$ (also, of $\mathfrak{G}$), and decompose $\mathfrak{K}$ and $\mathfrak{G}$ as the direct sum of $\mathfrak{H}$ and some root vector spaces (Cartan decompositions); then we have

$$\mathfrak{M}=\mathfrak{N}^+\oplus\mathfrak{N}^-;\ [\mathfrak{N}^\pm,\mathfrak{N}^\pm]=0,\ [\mathfrak{N}^+,\mathfrak{N}^-]=\mathfrak{K},$$

where $\mathfrak{N}^+=\sum_\alpha C E_\alpha$ and $\mathfrak{N}^-=\sum_\alpha CE_{-\alpha}$ (α runs through the so-called non-compact positive roots [4]). Such a decomposition will be given by

Proposition 2. $\mathfrak{N}^\pm=\{D_{x\bar{u}_2}\mp R_{xu_2}+D_{y\bar{u}_3}\pm R_{yu_3};\ x, y \in \mathfrak{C}\}$. *Hence, the complex vector spaces $\mathfrak{N}^+$ and $\mathfrak{N}^-$ are naturally isomorphic to the product space $\mathfrak{C}\times\mathfrak{C}$ respectively.*

Next, we have to determine a compact form $\mathfrak{G}_u$ of $\mathfrak{G}$ and the complex conjugation ι of $\mathfrak{G}$ with respect to $\mathfrak{G}_u$. For this, we denote by $\mathfrak{J}_R$, $\mathfrak{D}_R$ and $\mathfrak{R}_R$ the real forms of $\mathfrak{J}$, $\mathfrak{D}$ and $\mathfrak{R}$ respectively: namely $\mathfrak{J}_R$ is the exceptional simple Jordan algebra over the real Cayley numbers $\mathfrak{C}_R$, $\mathfrak{D}_R$ the derivation algebra of $\mathfrak{J}_R$ and $\mathfrak{R}_R$ the set of all translations of $\mathfrak{J}_R$ by elements with zero traces. Then we have

Proposition 3. $\mathfrak{G}_u=\mathfrak{D}_R\oplus\sqrt{-1}\,\mathfrak{R}_R=\mathfrak{K}_u\oplus\mathfrak{M}_u$, *where* $\mathfrak{K}_u=\mathfrak{G}_u\cap\mathfrak{K}$, $\mathfrak{M}_u=\mathfrak{G}_u\cap\mathfrak{M}$;

$$\mathfrak{K}_u=\mathfrak{D}_{0,R}\oplus\mathfrak{D}_{1,R}\oplus\sqrt{-1}\,\mathfrak{R}_{0,R}\oplus\sqrt{-1}\,\mathfrak{R}_{1,R}$$
$$\mathfrak{M}_u=\mathfrak{D}_{2,R}\oplus\mathfrak{D}_{3,R}\oplus\sqrt{-1}\,\mathfrak{R}_{2,R}\oplus\sqrt{-1}\,\mathfrak{R}_{3,R}.$$

Consequently, the corresponding non-compact form $\mathfrak{G}_R$, *which is dual to* $\mathfrak{G}_u$, *is given by*

$$\mathfrak{G}_R=\mathfrak{K}_u\oplus\sqrt{-1}\,\mathfrak{M}_u,$$

and we can define the complex conjugation ι *of* $\mathfrak{G}$ *over* $\mathfrak{G}_u$ *by the rule*

$$\iota(D_u+R_v)=D_{\tilde{u}}-R_{\tilde{v}},$$

where $\sim$ *is the complex conjugation of* $\mathfrak{C}$ *over* $\mathfrak{C}_R$.

4. Now, let $\tilde{\rho}$ denote the representation of Chevalley and Schafer as described above, and ρ_K the restriction of $\tilde{\rho}$ to $\mathfrak{K}$; then ρ_K is completely reducible and (ρ_K, V) $(V=\mathfrak{J})$ is decomposed into irreducible components (ρ_i, V_i) $(1\leq i\leq 3)$ as follows:

Proposition 4. $V=V_1\oplus V_2\oplus V_3$ $(\dim_c V_1=1,\ \dim_c V_2=16,\ \dim_c V_3=10)$, *where*

$$V_1=\{\xi e;\ \xi\in C\}$$
$$V_2=\{xu_2+yu_3;\ x, y\in\mathfrak{C}\}$$
$$V_3=\{\eta e_2+\zeta e_3+zu_1;\ \eta,\zeta\in C,\ z\in\mathfrak{C}\}:$$

in fact, ρ_1 *is the scaler representation determined by* $\rho_1(\sum R_{\xi_i e_i})=\xi_1$, ρ_2 *is equivalent to one of the half-spin representations of* $\mathfrak{o}(10, C)$ *and* ρ_3 *to the identity representation of* $\mathfrak{o}(10, C)$.

Moreover it can be easily proved the following

Proposition 5. $\mathfrak{R}^+(V_i)\subset V_{i-1}$, $\mathfrak{R}^-(V_{i-1})\subset V_i$ $(1\leq i\leq 3,\ V_0=\{0\})$, *and the highest weight of* $\tilde{\rho}$ *is given by the linear form* ξ_1 *on* $\mathfrak{R}_0(\subset\mathfrak{H})$.

Therefore the decomposition $V=V_1\oplus V_2\oplus V_3$ in Proposition 4 satisfies the condition in [5]; the arguments in [5] can be now applied to our case: namely, under the notation in [5], we see $p=n_1=1$, $r=n_2=16$, $n_3=10$, $q=26$, and our domain D has to be realized in $M_{p,r}=M_{1,16}$ ($M_{p,r}$ means the complex vector space of all (p,r)-matrices over C). On the other hand $M_{1,16}$ is now interpreted as the space of linear forms on V_2, where $V_2\cong\mathfrak{C}\times\mathfrak{C}$; so we identify $M_{1,16}$ with $\mathfrak{C}\times\mathfrak{C}$ in the canonical manner.

5. Denoting by z the pair $(x,y)\in\mathfrak{C}\times\mathfrak{C}$ of (complex) Cayley numbers, we write the corresponding elements of $\mathfrak{R}^+$ in Proposition 2 as $Z=\bar{E}_x^{(2)}+E_y^{(3)}$; $\bar{E}_x^{(2)}=D_{x\bar{u}_2}-R_{xu_2}$, $E_y^{(3)}=D_{y\bar{u}_3}+R_{yu_3}$, then, $Z^*=-\iota(Z)$ $(\in\mathfrak{R}^-)$ is given by

$Z^*=-E_{\tilde{x}}^{(2)}-\bar{E}_{\tilde{y}}^{(3)}$. ($\tilde{x}$ = the complex conjugate of x in $\mathfrak{C}$, etc.)

Moreover, the adjoint action $\theta[Z^*, Z]$ of the element $[Z^*, Z] \in \mathfrak{K}$ on $\mathfrak{N}^-$ is as follow (see [5]):

$$\theta[Z^*, Z] = \rho_2([Z^*, Z]) - \rho_1([Z^*, Z])$$

where $\rho_1([Z^*, Z]) = -Z_1Z_1^*$, $\rho_2([Z^*, Z]) = Z_1^*Z_1 - Z_2Z_2^*$ and $Z_1 \in M_{1,16}$, $Z_2 \in M_{16,10}$ are matrices determined by

$$Z = \begin{pmatrix} 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \\ 0 & 0 & 0 \end{pmatrix};$$

the left hand side, Z, is a matrix corresponding to the linear transformation $Z \in \mathfrak{N}^+$ with respect to certain fixed basis of V compatible with the decomposition in Proposition 4 (see [5]).

Now we introduce some notations concerning Cayley numbers: Let $\{c_0, c_1, \cdots, c_7\}$ denote the *canonical* basis of $\mathfrak{C}$ [8], for which we can write $x = \sum_{i=0}^7 x_i c_i$, $y = \sum_{i=0}^7 y_i c_i$ $(x_i, y_i \in C)$. To x and y corresponds the row vectors:

$$\boldsymbol{x} = \begin{pmatrix} x_0 \\ \vdots \\ x_7 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y_0 \\ \vdots \\ y_7 \end{pmatrix} \in M_{8.1} = C^8, \quad \boldsymbol{z} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in M_{16.1} = C^{16}.$$

Moreover, for any $x \in \mathfrak{C}$, B_x stands for the matrix $\in M_{8.8}$ corresponding to the linear transformation $y \to \overline{xy}$ $(y \in \mathfrak{C})$ of $\mathfrak{C}$ with respect to the basis $\{c_i\}$. Then we have

$$Z_1 = (-2{}^t\boldsymbol{x}, 2{}^t\boldsymbol{y}), \quad Z_2 = \begin{pmatrix} 0, & -\boldsymbol{x}, & B_y \\ \boldsymbol{y}, & 0, & -{}^t\bar{B}_{\bar{x}} \end{pmatrix}.$$

On the other hand, a result of Langlands [7], combined with our standpoint [5], asserts that our domain D is to be realized as the set of all pair $z = (x, y)$ such that the eigen-values of the hermitian operator $\theta[Z^*, Z]$ is smaller than 2: $\theta[Z^*, Z] < 2.\mathrm{I}$. Hence we obtain from the descriptions of above matrices the following result:

Theorem. *The irreducible bounded symmetric domain D of type (V) is realized as*

$$D = \left\{ z = (x, y) \in \mathfrak{C} \times \mathfrak{C}\,;\, {}^t\bar{\boldsymbol{z}}\boldsymbol{z} + \bar{\boldsymbol{z}}'{}^t\boldsymbol{z}' - \boldsymbol{z}^{(1)}{}^t\bar{\boldsymbol{z}}^{(1)} - \boldsymbol{z}^{(2)}{}^t\bar{\boldsymbol{z}}^{(2)} + \frac{1}{2} Z^t\bar{Z} < I \right\},$$

where

$$\boldsymbol{z}' = \begin{pmatrix} -\boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}, \quad Z = \begin{pmatrix} B_y \\ -{}^t\bar{B}_{\bar{x}} \end{pmatrix}, \quad \boldsymbol{z}^{(1)} = \begin{pmatrix} \boldsymbol{x} \\ 0 \end{pmatrix}, \quad \boldsymbol{z}^{(2)} = \begin{pmatrix} 0 \\ \boldsymbol{y} \end{pmatrix}.$$

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