51. On Some Homogeneous Boundary Value Problems Bounded Below

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§1. Introduction. Let Ω be a compact oriented Riemannian n-space with smooth boundary Γ. Let A be a linear partial differential operator on Ω of order $2m$. We assume A is strongly elliptic, that is, there is a constant $C>0$ such that, for any x in Ω and for any non zero vector ξ cotangent to Ω at x, we have

$$C^{-1}|\xi|^{2m}\leq \mathrm{Re}\,\sigma_{2m}(A)(x,\xi)\leq C|\xi|^{2m},$$

where $\sigma_{2m}(A)$ is the principal symbol of A. We consider normal systems $\{B_r\}_{r\in R}$, $R=(r_1,r_2,\cdots,r_m)$, of m boundary operators B_{r_j}. r_j is the order of B_{r_j}. We assume $r_j<2m$ for any $j=0,1,\cdots,m$. The problem to be considered is

Problem 1. Characterize those couples $\{A,\{B_r\}_{r\in R}\}$ which give, with some constants $1/2\geq\varepsilon\geq 0$, $C,\beta>0$, the estimate

$$\mathrm{Re}((A+\beta)u,u)_{L^2(\Omega)}\geq C\|u\|^2_{H^{m-\varepsilon}(\Omega)} \tag{1}$$

for all u in $H^{2m}_B(\Omega)=\{u\in H^{2m}(\Omega)\,;\,B_r u|_\Gamma=0,\text{ for any } r\in R\}$.

Here $H^s(\Omega)$ denotes the Sobolev space on Ω of order s, $\|\ \ \|_{H^s(\Omega)}$ is its norm and $(\ ,\ )_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)$.

If $1/2>\varepsilon\geq 0$, the problem was treated in far stronger form in [3]. In this note we concern with the case $\varepsilon=1/2$. So the problem is

Problem 1′. Characterize those couples $\{A,\{B_r\}_{r\in R}\}$ which give, with some constants $C,\beta>0$, the estimate

$$\mathrm{Re}((A+\beta)u,u)_{L^2(\Omega)}\geq C\|u\|^2_{H^{m-1/2}(\Omega)} \tag{2}$$

for all u in $H^{2m}_B(\Omega)$.

We assume the following hypothesis (H) that was proved in the case $0\leq\varepsilon<1/2$ necessary for the estimate (1) to hold. (See [3] and [6].)
(H) The set R coincides with one of the R_j's defined by $R_j=(0,1,\cdots\cdots,m-j-1,m,m+1,\cdots,m+j-1)$, $1\leq j\leq m$. Under this hypothesis we give a necessary and sufficient condition for the estimate (2) to hold.

Proofs are omitted. Detailed discussions will be published elsewhere.*)

§2. Results. We denote by ν the interior unit normal to Γ and

*) This work was done during the author's stay in Paris. He expresses his hearty thanks to Professor J. L. Lions for his constant encouragement.

by D_n the normal derivative $-i\frac{\partial}{\partial \nu}$ multiplied by $-i=-\sqrt{-1}$. S_j is the complement of R_j in the set $\{0, 1, 2, \cdots, 2m-1\}$. Then B_r, $r \in R_j$ can be written as

$$B_r = D_n^r - \sum_{\substack{\rho \in S_j \\ \rho < r}} B_{r-\rho}^r D_n^\rho,$$

where $B_{r-\rho}^r$ is a pseudo-differential operator on Γ of order $\leq r-\rho$. Let $\Lambda=(1-\Delta')^{1/2}$ where Δ' is the Laplace-Beltrami operator associated with the metric on Γ. Then Λ^k is an isomorphism from $H^s(\Gamma)$ to $H^{s-k}(\Gamma)$. A^* denotes the formal adjoint of A.

We choose and fix α so large that we can solve uniquely the problem:

$$\begin{gathered}(A+A^*+2\alpha)v=0 \\ D_n^k v|_\Gamma = \Lambda^k \phi_k, \quad m-1 \geq k \geq m-j, \\ D_n^k v|_\Gamma = 0, \quad m-j-1 \geq k \geq 0,\end{gathered}$$

and obtain the estimates, for any $s \in \boldsymbol{R}$,

$$(5) \qquad C^{-1} \sum_{k=m-j}^{m-1} \|\phi_k\|^2_{H^{s-1/2}(\Gamma)} \leq \|v\|^2_{H^s(\Omega)} \leq C \sum_{k=m-j}^{m-1} \|\phi_k\|^2_{H^{s-1/2}(\Gamma)}.$$

Here and hereafter we denote by C different constants >0 in different occurrences.

Now we fix $B=\{B_r\}_{r \in R_j}$. We decompose any u in $H_B^{2m}(\Omega)$ into sum of two functions v and w:

$$(6) \qquad u = v + w,$$

where

$$(7) \qquad (A+A^*+2\beta)v=0 \text{ on } \Omega, \quad D_n^k v|_\Gamma = D_n^k u|_\Gamma, \quad 0 \leq k \leq m-1,$$

and $D_n^k w|_\Gamma = 0$, $0 \leq k \leq m-1$. This implies that $D_n^k v|_\Gamma = 0$ for $0 \leq k \leq m-j-1$. We set $D_n^k u|_\Gamma = \Lambda^k \varphi_k$, $m-j \leq k \leq m-1$. Let $H_B^s(\Omega)$ be the closure of $H_B^{2m}(\Omega)$ in $H^s(\Omega)$. Then $H_B^m(\Omega) = \{u \in H^m(\Omega) : D_n^k u|_\Gamma = 0,\ 0 \leq k \leq m-j-1\}$. The decomposition (6) is a topological decomposition of $H_B^m(\Omega)$. (See [5].) Now we take any u in $H_B^{2m}(\Omega)$. Then using the boundary condition $B_r u|_\Gamma = 0$ and the decomposition (6), we can find pseudo-differential operators $H_{p,q}$ on Γ of order $2m-1$, $m-j \leq p$, $q \leq m-1$, such that

$$(8) \qquad \begin{aligned}&\operatorname{Re}((A+\beta)u, u)_{L^2(\Omega)} \\ &= \operatorname{Re}((A+\beta)w, w)_{L^2(\Omega)} + \sum_{p,q=m-j}^{m-1} (H_{pq}(\beta)\varphi_q, \varphi_p)_{L^2(\Gamma)}.\end{aligned}$$

(See [2].)

Let T be the 1 dimensional circle$=\boldsymbol{R}/2\pi\boldsymbol{Z}$. We consider the elliptic operator $\tilde{A}=A+D_s^{2m}$, $s \in T$, on $\Omega \times T$ and boundary operators $\{B_r\}_{r \in R_j}$ on $\Gamma \times T$. $H_B^s(\Omega \times T)$ denotes the closure in $H^s(\Omega \times T)$ of $H_B^{2m}(\Omega \times T) = \{f \in H^{2m}(\Omega \times T) : B_r f|_{\Gamma \times T} = 0,\ r \in R_j\}$. Decomposition corresponding to (6) holds for functions in $H_B^{2m}(\Omega \times T)$, that is, for any f in $H_B^{2m}(\Omega \times T)$,

$$(9)\qquad f=g+h,\quad (\tilde{A}+\tilde{A}^*+2\beta)g=0 \text{ on } \Omega\times T,$$
$$D_n^k g|_{\Gamma\times T}=D_n^k f|_{\Gamma\times T},\quad 0\le k\le m-1.$$

We set $D_n^k f|_{\Gamma\times T}=\tilde{\Lambda}^k\phi_k$, $m-j\le k\le m-1$, where $\tilde{\Lambda}=(1-\Delta'+D_s^2)^{1/2}$. Just as we did above, we can find pseudo-differential operators $\tilde{H}_{pq}(\beta)$ on $\Gamma\times T$ of order $2m-1$ such that for any f in $H_B^{2m}(\Omega\times T)$

$$(10)\qquad \mathrm{Re}((\tilde{A}+\beta)f,f)_{L^2(\Omega\times T)} = \mathrm{Re}((\tilde{A}+\beta)h,h)_{L^2(\Omega\times T)}+\sum_{p,q=m-j}^{m-1}(\tilde{H}_{pq}(\beta)\phi_q,\phi_p)_{L^2(\Gamma\times T)}.$$

Our first result is

Theorem 1. *Each of the following four propositions are equivalent to the other*:

(i) *There are some β_1, $C_1>0$ such that the estimate* (2) *holds for any $u\in H_B^{2m}(\Omega)$.*

(ii) *There are some β_2, $C_2>0$, such that the estimate*

$$(11)\qquad \mathrm{Re}((\tilde{A}+\beta_2)f,f)_{L^2(\Omega\times T)}\ge C_2\|f\|^2_{H^{m-1/2}(\Omega\times T)}$$

holds for any f in $H_B^{2m}(\Omega\times T)$.

(iii) *There are some constants β_3, $C_3>0$ such that the estimate*

$$(12)\qquad \sum_{p,q=m-j}^{m-1}(H_{pq}(\beta_3)\varphi_q,\varphi_p)_{L^2(\Gamma)}\ge C_3\sum_{p=m-j}^{m-1}\|\varphi_p\|^2_{H^{m-1}(\Gamma)}$$

holds for any $\varphi_{m-j},\varphi_{m-j+1},\cdots,\varphi_{m-1}\in H^{m-1/2}(\Gamma)$.

(iv) *There are some constants $\gamma,\beta_4,C_4>0$ such that the estimate*

$$(13)\qquad \sum_{p,q=m-j}^{m-1}(\tilde{H}_{pq}(\gamma)\phi_q,\phi_p)_{L^2(\Gamma\times T)}+\beta_4\sum_{p=m-j}^{m-1}\|\phi_p\|^2_{H^{-1/2}(\Gamma\times T)} \ge C_4\sum_{p=m-j}^{m-1}\|\phi_p\|^2_{H^{m-1}(\Gamma\times T)}$$

holds for any $\phi_{m-j},\phi_{m-j+1},\cdots,\phi_{m-1}$ in $H^{m-1/2}(\Gamma\times T)$.

Remark 1. In the case $0\le\varepsilon<1/2$ the estimate holds with some β, $C>0$, if and only if, with some γ, β, $C>0$, the estimate

$$(14)\qquad \sum_{p,q=m-j}^{m-1}(H_{pq}(\gamma)\varphi_q,\varphi_p)_{L^2(\Gamma)}+\beta\sum_{p=m-j}^{m-1}\|\varphi_p\|^2_{H^{-1/2}(\Gamma)} \ge C\sum_{p=m-j}^{m-1}\|\varphi_p\|^2_{H^{m-1/2-\varepsilon}(\Gamma)}$$

holds for any $\varphi_{m-j},\cdots,\varphi_{m-1}$ in $H^{m-1/2}(\Gamma)$.

We consider pseudo-differential operators $\tilde{H}_{pq}(\gamma)$, $m-j\le p$, $q\le m-1$, of order $2m-1$ defined on $\Gamma\times T$ and satisfying the property (iv) of Theorem 1.

The property (iv) of Theorem 1 can be localized.

Theorem 2. *Assume that there exists a family of finite number of real functions $\{\mu_k(x)\}_{k=1}^N$ in $\mathcal{D}$ $(\Gamma\times T)$ satisfying*

(i) $\sum\mu_k(x,s)^2=1$,

(ii) *for any $\phi_{m-j},\phi_{m-j+1},\cdots,\phi_{m-1}\in\mathcal{D}$ $(\Gamma\times T)$ and for any k the following estimate holds*:

$$(15)\qquad \sum_{p,q=m-j}^{m-1}(\tilde{H}_{pq}(\gamma)\mu_k\phi_q, \mu_k\phi_p)_{L^2(\Gamma\times T)}+\beta\sum_{p=m-j}^{m-1}\|\mu_k\phi_p\|^2_{H^{-1/2}(\Gamma\times T)} \geq C\sum_{p=m-j}^{m-1}\|\mu_k\phi_p\|^2_{H^{m-1}(\Gamma\times T)}.$$

Then for any $\phi_{m-j}, \phi_{m-j+1}, \cdots, \phi_{m-1} \in \mathcal{D}(\Gamma\times T)$ *the estimate* (13) *holds with some* β_4, C_4 *and* $\gamma_4>0$.

Let Ω be any open set (not necessarily connected) in $\boldsymbol{R}^n$. Let Q_{rs}, $m-j\leq r, s\leq m-1$, be pseudo-differential operators of order 1 defined in Ω. $q_{rs}(x,\xi)\sim\sum_{j=0}^{\infty}q^j_{rs}(x,\xi)$ denote the symbol of Q_{rs}. We assume the matrix $(q^0_{rs}(x,\xi))_{rs}$ of the principal symbols of Q_{rs} is Hermitian. Then we have

Theorem 3. *The following two properties are equivalent:*

(i) *For any compact set K in Ω, there are constants C_0 and $C_1>0$ such that, for any* $\phi_{m-j}, \phi_{m-j+1}, \cdots, \phi_{m-1}\in\mathcal{D}(K)$,

$$(16)\quad \mathrm{Re}\sum_{r,s=m-j}^{m-1}(Q_{rs}\phi_s, \phi_r)_{L^2(\Omega)}+C_1\sum_{r=m-j}^{m-1}\|\phi_r\|^2_{H^{-1/2}(\Omega)}\geq C_0\sum_{r=m-j}^{m-1}\|\phi_r\|^2_{H^0(\Omega)}.$$

(ii) *For any compact set K_1 in Ω, there exist constant $C>0$, integer $N>0$ and a function $\varepsilon(\xi)$ with $\varepsilon(\xi)\to 0$ when $|\xi|\to\infty$ such that, for any $x\in K_1$,* $\psi_{m-j}, \cdots, \psi_{m-1}\in\mathcal{D}(\boldsymbol{R}^n)$,

$$(17)\quad \mathrm{Re}\sum_{r,s=m-j}^{m-1}\ \sum_{|\alpha|+|\beta|\leq 2}\frac{|\xi|^{(|\beta|-|\alpha|)/2}}{\alpha!\beta!}q^{0(\beta)}_{rs(\alpha)}(x,\xi)\int_{\boldsymbol{R}^n}(iD_y)^\beta\psi_s(y)\overline{(-iy)^\alpha\psi_r(y)}dy$$
$$+\mathrm{Re}\sum_{r,s=m-j}^{m-1}q^1_{rs}(x,\xi)\int_{\boldsymbol{R}^n}\psi_s(y)\overline{\psi_r(y)}dy$$
$$+\varepsilon(\xi)\sum_{|\alpha|+|\beta|\leq N}\ \sum_{r=m-j}^{m-1}\int_{\boldsymbol{R}^n}|D^\alpha_y y^\beta\psi_r(y)|dy$$
$$\geq C\sum_{r=m-j}^{m-1}\int_{\boldsymbol{R}^n}|\psi_r(y)|^2dy,$$

where $q^{0(\beta)}_{(\alpha)}(x,\xi)=D^\alpha_x D^\beta_\xi q^0(x,\xi)$.

Remark 2. The estimate (14) holds for any $\varphi_{m-j}, \cdots, \varphi_{m-1}\in H^{m-1/2}(\Gamma)$ if and only if the matrix defined by the principal symbols $\sigma_{2m-1}(H_{pq}(\beta))(x',\xi')$ is uniformly positive definite. Thus we can prove the result in [3] without the assumption that $\sigma_{2m}(A)(x,\xi)$ is real.

To prove Theorem 3 we use the following theorem which is interesting in itself.

Theorem 4.*) *Let K be any compact set in an open set Ω in $\boldsymbol{R}^n$ and let P be a peudo-differential operator of order ρ defined on Ω, whose symbol is denoted by $p(x,\xi)$. Assume $\varphi\in\mathcal{D}(\Omega)$ is identically 1 in some neighbourhood of K. Then for any $N>0$, there is a constant $C>0$ such that for any $x\in K$, $\xi\in\boldsymbol{R}^n$ with $|\xi|\geq 1$, and ϕ, φ in $\mathcal{D}(\boldsymbol{R}^n)$,*

*) During the preparation of this article the author had a chance to know that A. P. Calderón also had obtained, independently, a result similar to Theorem 4 in a little stronger form.

$$|\xi|^{n/2}\int_{\Omega}(P\varphi v_1)(y)\overline{\varphi v_2(y)}dy$$
$$-\sum_{|\alpha|,|\beta|<N}\frac{|\xi|^{(|\beta|-|\alpha|)/2}}{\alpha!\beta!}p^{(\beta)}_{(\alpha)}(x,\xi)\int(iD_y)^{\beta}\psi(y)\overline{(-iy)^{\alpha}\phi(y)}dy$$
$$\leq C|\xi|^{-N/2+2|\beta|+n}\|\psi\|_{H^{3N/2}}\|(1+|y|)^N\phi\|_{H^0(\boldsymbol{R}^n)},$$

where $v_1(y)=\psi((y-x)|\xi|^{1/2})e^{iy\cdot\xi}$ *and* $v_2(y)=\phi((y-x)|\xi|^{1/2})e^{iy\cdot\xi}$.

Proofs of Theorems 3 and 4 are omitted here. They are similar to the discussion in [4].

References

[1] Agmon, S.: The coerciveness problem for integro-differential forms. J. Analyse Math., **6**, 183-223 (1958).
[2] Fujiwara, D.: On the asymptotic behaviour of the Green operators for elliptic boundary problems and the pure imaginary powers of some second order operators (to appear in J. Fac. Sci. Univ. Tokyo).
[3] Fujiwara, D., and Shimakura, N.: Sur les problèmes aux limites elliptiques stablement variationnels (to appear).
[4] Hörmander, L.: Pseudo-differential operators and non elliptic boundary problems. Ann. of Math., **83**, 129-209 (1966).
[5] Lions, J. L., and Magenes. E.: Problèmes aux limites non homogènes et applications, Vol. 1. Dunod, Paris (1968).
[6] Shimakura, N.: Problèmes aux limites variationnels du type elliptique (to appear in Ann. Sci. Ecole Norm. Sup. de Paris).