# 49. *Semi-groups of Nonlinear Operators on Closed Convex Sets*

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Let $H$ be a real or complex Hilbert space, whose inner product and norm are denoted by $\langle\ ,\ \rangle$ and $|\ \ |$. Let $D\subseteq H$ and let $\{T_t;\ t\geq 0\}$ be a semi-group of mappings of $D$ into itself. For $K\subseteq H$ and $x\in H$, let $C_K(x)$ be the strong closure of the set $\{r(z-x);\ r\geq 0,\ z\in K\}$. If $K$ is a closed subset of $D$ and if $T_tK\subseteq K$ for $t\geq 0$, it is obvious that we have

$$A_0x=s-\lim_{h\downarrow 0}\frac{1}{h}(T_hx-x)\in C_K(x),\ x\in\partial K \tag{1}$$

whenever the strong limit exists. We are interested in the converse problem: under what conditions on $\{T_t\}$ and $K$ does (1) imply $T_tK\subseteq K$, $t\geq 0$. We consider this converse problem when $\{T_t\}$ is a strongly continuous semi-group of contractions and $K$ is a closed convex set.[1] Our result is given in Theorem 1 below. Theorem 1 enables us to prove Theorem 2 which gives a sufficient condition on a (nonlinear) dissipative operator that it be a strong generator of a semi-group of contractions on a closed convex set. Finally we apply Theorem 2 to prove Theorem 3 which shows the existence of a sequence of semi-groups with continuous infinitesimal generators approximating a given semi-group of contractions on a closed convex set.

We begin with a simple lemma which we shall need later.

**Lemma 1.** *Let $u\in H$ and $K$ be a closed convex subset of $H$. Then there exists one and only one element $v\in K$ such that $|u-v|=\inf\{|u-z|;\ z\in K\}$. Moreover, we have the inequality*

$$\mathrm{Re}\,\langle y,u-v\rangle\leq 0,\ y\in C_K(v). \tag{2}$$

**Proof.** We shall prove the inequality (2). For $y\in C_K(v)$ we can find a sequence $r_n\geq 0$ and a sequence $z_n\in K$ such that $y=s-\lim_{n\to\infty} r_n(z_n-v)$. Since $v+c(z_n-v)\in K$ for $0\leq c\leq 1$, it follows that

$$|u-v-c(z_n-v)|\geq|u-v|,\qquad 0\leq c\leq 1.$$

This gives

$$\mathrm{Re}\,\langle z_n-v,u-v\rangle\leq 0.$$

1) It is to be noted that (1) does not necessarily imply $T_tK\subseteq K$, $t\geq 0$, if we do not assume that $T_t$ is a contraction for $t\geq 0$.

Hence we have

$$\mathrm{Re}\,\langle y,\, u-v\rangle=\lim_{n\to\infty}\mathrm{Re}\,\langle r_n(z_n-v),\, u-v\rangle\leq 0.$$

**Theorem 1.** *Let $C$ be a subset of $H$, and $K$ a closed convex subset of $C$. Let $\{T_t\,;\,t\geq 0\}$ be a strongly continuous semi-group of contractions on $C$ with strong generator $A_0$. If $\partial K\subseteq D(A_0)$ and $A_0x\in C_K(x)$ for all $x\in\partial K$, then we have $T_tx\in K$ for all $x\in K$ and all $t\geq 0$.*

**Proof.** Suppose that there is an $x_1\in K$ such that $T_bx_1\in C\backslash K$ for some $b>0$. Then there is an $a\in[0, b)$ such that $T_ax_1\in\partial K$ and $T_tx_1\in C\backslash K$ whenever $t\in(a, b]$, since $T_tx_1$ is continuous in $t$ and $K$ is closed. Let $x_0=T_ax_1$ and $\delta=b-a$. Thus $\delta>0$. We set $u(t)=T_tx_0$.

Since $K$ is closed and convex, there exists, for $0\leq t\leq\delta$, a unique element $v(t)\in\partial K$ satisfying $|u(t)-v(t)|=\inf\{|u(t)-z|\,;\,z\in K\}$. We define $\rho(t)=|u(t)-v(t)|$. It is clear that $\rho(0)=0$ and $\rho(t)>0$ whenever $0<t\leq\delta$. Since $\rho(s)\leq|u(s)-v(t)|$ for $s,\,t\geq 0$, we find

$$\rho(s)-\rho(t)\leq|u(s)-v(t)|-|u(t)-v(t)|\leq|u(s)-u(t)|.$$

Hence we obtain

$$|\rho(s)-\rho(t)|\leq|u(s)-u(t)|\qquad\text{for } s,\,t\geq 0. \tag{3}$$

Since $x_0\in D(A_0)$, $u(t)$ is in $D(A_0)$ for $t\geq 0$ and $u$ is Lipschitz continuous (see [1], Lemma 1.1). Hence, by (3), $\rho$ is Lipschitz continuous and, since $\rho(0)<\rho(\delta)$, there is a $t^*\in(0,\delta)$ such that $u$ and $\rho$ are differentiable at $t^*$ and $\frac{d\rho}{dt}(t^*)>0$. We now define $r(t)=|u(t)-v(t^*)|$ for $t\geq 0$. It is clear that $\rho(t^*)=r(t^*)$ and $\rho(t)\leq r(t)$ for $t\geq 0$. Therefore, we have

$$\frac{dr^2}{dt}(t^*)=\frac{d\rho^2}{dt}(t^*)=2\rho(t^*)\frac{d\rho}{dt}(t^*)>0. \tag{4}$$

Using Lemma 1 and the dissipativity of $A_0$, we obtain

$$\begin{aligned}\frac{dr^2}{dt}(t^*)&=2\,\mathrm{Re}\,\langle A_0u(t^*),\;\; u(t^*)-v(t^*)\rangle\\ &\leq 2\,\mathrm{Re}\,\langle A_0u(t^*)-A_0v(t^*),\, u(t^*)-v(t^*)\rangle\leq 0,\end{aligned}$$

which contradicts the inequality (4). The proof of Theorem 1 is complete.

For $x,\,y\in H$, we denote by $(x,\,y)$ an element of the Cartesian product $H\times H$. If $A\subseteq H\times H$, then $D(A)$ denotes the set of all $x\in H$ such that $(x,\,y)\in A$ for some $y\in H$. For $x\in D(A)$, $Ax$ denotes the set of all $y\in H$ such that $(x,\,y)\in A$. If $Ax$ consists of a single element for all $x\in D(A)$, then $A$ is called a function. A subset $A$ of $H\times H$ is said to be dissipative if for all $(x_1,\,y_1)$ and $(x_2,\,y_2)$ in $A$

$$\mathrm{Re}\,\langle x_1-x_2,\, y_1-y_2\rangle\leq 0.$$

**Definition.** Let $C$ be a subset of $H$ and $A$ a dissipative subset of $C\times H$. $A$ is said to be

(a) *$C$-maximal dissipative* if $A$ is not properly contained in any dissipative subset of $C\times H$;

(b) *functionally $C$-maximal dissipative* if $A$ is a function which is not properly contained in any dissipative function with domain in $C$.

If $A$ is a $C$-maximal dissipative set, then for $x\in D(A)$ we have

$$Ax=\{y\in H;\ \mathrm{Re}\langle x'-x,\ y'-y\rangle\leq 0 \qquad \text{for all } (x',y')\in A\}$$

which is a closed convex set. Hence there is a unique element $A^{0}x\in Ax$ such that $|A^{0}x|\leq|y|$ for all $y\in Ax$. Thus we have defined the function $A^{0}$ with domain $D(A^{0})=D(A)$, which will be called the minimal cross-section of $A$ (see [1]).

**Example.** Let $H=R^{2}$. Let $x_1=(1,1)$, $x_2=(-1,1)$, $x_3=(-1,-1)$, $x_4=(1,-1)$ and $C=\{x_1,x_2,x_3,x_4\}\subset H$. For $h\geq 0$, we define

$$A_hx_1=(-\infty,-h]\times(-\infty,h],\qquad A_hx_2=[-h,\infty)\times(-\infty,-h],$$
$$A_hx_3=[h,\infty)\times[-h,\infty)\quad\text{and}\quad A_hx_4=(-\infty,h]\times[h,\infty).$$

Then $A_h$ is $C$-maximal dissipative subset of $H\times H$ and we have $A_h^{0}x_1=(-h,0)$, $A_h^{0}x_2=(0,-h)$, $A_h^{0}x_3=(h,0)$ and $A_h^{0}x_4=(0,h)$. Hence, if $h>k\geq 0$, then $A_h^{0}\subset A_k$ and $A_h^{0}\neq A_k^{0}$ (cf., [1], Theorem 2.4 (b)).

The next theorem gives a sufficient condition on a dissipative operator that it be the strong generator of a semigroup of contractions on a closed convex set.

**Theorem 2.** *Let $K$ be a closed convex subset of $H$ and let $A$ be a subset of $K\times H$ with the following properties:*

( i ) *$A$ is $K$-maximal dissipative and $A^{0}$ is functionally $K$-maximal dissipative;*

( ii ) *$D(A)$ is dense in $K$;*

(iii) *$\partial K\subseteq D(A)$ and $A^{0}x\in C_K(x)$ for all $x\in\partial K$.*

*Then there exists a uniquely determined strongly continuous semi-group of contractions on $K$ such that its strong generator is $A^{0}$.*

**Proof.** Let $B$ be an $H$-maximal dissipative set containing $A$. Then there is a strongly continuous semi-group $S$ of contractions on $D(B)$ such that its strong generator is $B^{0}$(see [1], Theorem 1). Let $T=\{T_t;\ t\geq 0\}$ be a strongly continuous semi-group of contractions which is a maximal extension of $S$. Then it is clear that the domain $C$ of $T$ is closed. Therefore, condition (ii) together with $D(A)\subseteq C$ will imply $K\subseteq C$. Let $G$ be the strong generator of $T$ so that $G$ is a dissipative extension of $B^{0}$. Since $A$ is a $K$-maximal dissipative subset of $B$, $B^{0}$ is an extension of $A^{0}$. Thus $G$ is an extension of $A^{0}$. Now it follows from (i) that $A^{0}$ is the restriction of $G$ to $K\cap D(G)$. Hence, by (iii) and Theorem 1, we have $T_tK\subseteq K$ for all $t\geq 0$. We have proved the existence of a strongly continuous semi-group on $K$ with strong generator $A^{0}$.

To prove the uniqueness, let $U=\{U_t; t\geq 0\}$ and $V=\{V_t; t\geq 0\}$ be two strongly continuous semi-groups of contractions on $K$ such that $A^0$ is the strong generator of $U$ as well as of $V$. Then, for every $x\in D(A)$, $U_tx$ and $V_tx$ are absolutely continuous on $[0,\infty)$ and we have $U_tx\in D(A)$, $V_tx\in D(A)$ for all $t\geq 0$ and $\frac{d}{dt}U_tx=A^0U_tx$, $\frac{d}{dt}V_tx=A^0V_tx$ for almost all $t\geq 0$ (see [1], Theorem 1.4). Therefore, it follows from the dissipativity of $A^0$ that $U_tx=V_tx$ for $x\in D(A)$ and $t\geq 0$. Hence, by (ii), we have $U_tx=V_tx$ for all $x\in K$ and all $t\geq 0$.

**Remarks.** (a) In Theorem 2, (i) and (iii) do not imply (ii) generally. For example, let $H=R^1$, $K=[0,\infty)$ and $f$ be a real-valued, continuous and decreasing function defined on $[0, 1)$ such that $f(0)\geq 0$ and $\lim_{x\uparrow 1} f(x)=-\infty$. Let $A$ be a subset of $K\times H$ defined as follows:

$$Ax=\begin{cases}[f(0),\infty) & \text{if } x=0,\\ \{f(x)\} & \text{if } 0<x<1,\\ \phi(\text{empty set}) & \text{if } x\geq 1.\end{cases}$$

Then $A$ satisfies (i) and (iii) in Theorem 2. However, $D(A)$ is not dense in $K$.

(b) In Theorem 2, if $K$ is the closed convex hull of $\partial K$, then (i) and (iii) imply (ii). This implication is proved as follows. In the proof of Theorem 2, we showed the existence of a semi-group on $K$ with strong generator $A^0$ using (ii) once for all to obtain $K\subseteq C$. We note that $C$ is closed and convex (see [5]). Therefore, if $K$ is the closed convex hull of $\partial K$, $\partial K\subseteq D(A)\subseteq C$ implies $K\subseteq C$. Hence there is a semi-group on $K$ such that its strong generator is $A^0$. Then $D(A^0)$ is dense in $K$ (see [5], [3]).

As a simple application of Theorem 2 we shall prove the following theorem.

**Theorem 3.** *Let $K$ be a closed convex subset of $H$ and let $\{T_t; t\geq 0\}$ be a weakly continuous semi-group of contractions on $K$. For a fixed positive real number $h$, we define*

$$A_hx=\frac{1}{h}(T_hx-x),\qquad x\in K.$$

*Then there is a strongly continuous semi-group of contractions on $K$ such that its strong generator is $A_h$.*

The proof of Theorem 3 is based on the following lemma.

**Lemma 2.** *Let $K\subseteq H$ and let $A$ be a dissipative function defined on $K$ satisfying the following conditions:*

(i) *There is a positive real number $r$ such that*

$$x+cAx\in K\qquad \text{for } x\in K \text{ and } 0\leq c\leq r,$$

(ii) $w-\lim_{c\downarrow 0} A(x+cAx)=Ax$ *for* $x\in K$.

Then, for any dissipative set $B$ containing $A$, we have

$$|Ax| \leq |y| \qquad \text{for } x \in K \text{ and } y \in Bx.$$

**Proof.** Let $x \in K$ and $y \in Bx$. We set

$$y_c = A(x + cAx) \qquad \text{for } 0 < c \leq r.$$

It follows from $A \subseteq B$ and the dissipativity of $B$ that

$$\mathrm{Re}\,\langle y_c - y, Ax\rangle = \frac{1}{c}\,\mathrm{Re}\,\langle y_c - y, (x + cAx) - x\rangle \leq 0.$$

Then we have

$$\mathrm{Re}\,\langle |Ax|\,y_c - |y|\,Ax, Ax\rangle \leq \mathrm{Re}\,\langle |Ax|\,y - |y|\,Ax, Ax\rangle \leq 0.$$

Therefore it follows from (ii) that

$$|y|\,|Ax|^2 \geq \lim_{c \downarrow 0} \mathrm{Re}\,\langle |Ax|\,y_c, Ax\rangle = |Ax|^3.$$

**Proof of Theorem 3.** $A_h$ is a dissipative function on $K$ satisfying the conditions (i) and (ii) in Lemma 2 with $r = h$. Let $B$ be a $K$-maximal dissipative set containing $A_h$. Then it follows from Lemma 2 that $B^0 x = A_h x$ for $x \in K$. Hence we have $B^0 x \in C_K(x)$ for $x \in K$. Since $D(B) = K$, the proof of Theorem 3 is complete by Theorem 2.

## References


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