83. On Semilattices of Groups

By S. LAJOS
K. Marx University of Economics, Budapest, Hungary

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Let S be a semigroup. Following the notation and terminology of A. H. Clifford and G. B. Preston's monograph [1] we shall say that S is a semilattice of groups if S is a set-theoretical union of a set $\{G_\alpha, \alpha \in I\}$ of mutually disjoint subgroups G_α such that, for every α, β in I, the products $G_\alpha G_\beta$ and $G_\beta G_\alpha$ are both contained in the same G_γ ($\gamma \in I$).

Recently the author proved the following ideal-theoretical characterizations of semigroups which are semilattices of groups.[1]

Theorem 1. *For a semigroup S the following conditions are mutually equivalent*:

(A) *S is a semilattice of groups.*

(B) $L \cap R = LR$ *for any left ideal L and for any right ideal R of S.*

(C) $L_1 \cap L_2 = L_1 L_2$ *and* $R_1 \cap R_2 = R_1 R_2$ *for any left ideals L_1, L_2 and for any right ideals R_1, R_2 of S, respectively.*

(D) $A \cap L = LA$ *and* $A \cap R = AR$ *for any left ideal L, for any right ideal R and for any two-sided ideal A of S.*

Now we establish some further criteria for an arbitrary semigroup to be a semilattice of groups. Namely we prove the result as follows.

Theorem 2. *The assertions* (E)–(H) *are equivalent with each other and with any of the conditions* (A)–(D) *of Theorem* 1:

(E) $Q_1 \cap Q_2 = Q_1 Q_2$ *for any two quasi-ideals Q_1, Q_2 of S.*

(F) $B \cap Q = BQ$ *for any bi-ideal B and for any quasi-ideal Q of S.*

(G) $B \cap Q = QB$ *for any bi-ideal B and for any quasi-ideal Q of S.*

(H) $B_1 \cap B_2 = B_1 B_2$ *for any two bi-ideals B_1, B_2 of S.*

Proof. It is easy to see that the conditions (E)–(H) are sufficient to insure that S be a semilattice of groups. Conversely if S is a semilattice of groups then every one-sided ideal of S is two-sided. Since every quasi-ideal Q of an arbitrary semigroup S can be represented as the intersection of a left ideal of S and a right ideal of S we obtain that Q is also a two-sided ideal of S. Similarly each bi-ideal of S is also a two-sided ideal of S. Then any of the relations (E)–(H) follows from $A \cap B = AB$ (A, B are arbitrary two-sided ideals of S) which is a consequence of the regularity of S.

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1) See the author's papers [2], [3], and [4].

We remark that the following result is also true and the proof is similar to that of Theorem 2.

Theorem 3. *For a semigroup S the following conditions are equivalent with each other and with any of the assertions* (A)–(H) *of Theorem* 1 *and Theorem* 2.

(I) $A_1 \cap A_2 = A_1 A_2$ *for any two* (m, n)*-ideals of* S, *where* $m>0$, $n>0$.

(J) $B_1 \cap B_2 = B_1 B_2$ *for any two* $(m, 0)$*-ideals and for any two* $(0, n)$*-ideals of* S.

(K) $A \cap B = AB$ *for any* $(0, n)$*-ideal* A *of* S *and for any* $(m, 0)$*-ideal* B *of* S.

For the definition of (m, n)-ideal we refer to the author's paper [5].

References

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