80. *Notes on Generalized Commuting Properties of Skew Product Transformations*

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1. Introduction. Let (M, Σ, m) be a measure space where M is a set of elements, Σ a σ-field of measurable subsets of M, and m a countably additive measure on Σ. An invertible measure-preserving transformation T of the measure space (M, Σ, m) is a one-to-one mapping of M onto itself such that if $B \in \Sigma$ then TB and $T^{-1}B \in \Sigma$ with $m(TB)=m(T^{-1}B)=m(B)$. Let $\mathfrak{G}$ be the group of all invertible measure-preserving transformations of (M, Σ, m) with I denoting the identity transformation on M. Associated with $T \in \mathfrak{G}$ is a sequence $C_n(T)$, $n=0,1,2,\cdots$, of subfamilies of $\mathfrak{G}$ defined inductively as follows:

$$C_o(T)=\{S \in \mathfrak{G} \mid S=I \text{ a.e.}\},$$
$$C_n(T)=\{S \in \mathfrak{G} \mid STS^{-1}T^{-1} \in C_{n-1}(T)\}.$$

It is clear that $C_n(T) \subset C_{n+1}(T)$ for each n. If there exists an integer N such that $C_N(T)=C_{N+1}(T)$ then $C_n(T)=C_N(T)$ for all $n \geqq N$. R. L. Adler [1] called $C_n(T)$ the nth class of generalized T-commuting transformations and defined the generalized commuting order $N(T)$ of T as follows:

$$N(T)=\begin{cases} \min\{n \mid C_n(T)=C_{n+1}(T)\} \text{ if there exists an integer } N \text{ such} \\ \text{that } C_N(T)=C_{N+1}(T), \\ \infty \text{ if } C_n(T) \neq C_{n+1}(T) \text{ for each } n. \end{cases}$$

Let H be the two-dimensional torus, i.e., $H=K \times K$, where $K=\{\exp[2\pi i t] \mid 0<t \leqq 1\}$, equipped with the normalized Haar measure λ and let $T_{\gamma,\mu}$ denote the invertible measure-preserving transformation on H which is defined by

$$T_{\gamma,\mu}: (x, y) \to (x\gamma, y \cdot x^{\mu})$$

where γ is an element of K such that $\gamma^n \neq 1$ for every $n \neq 0$ and μ a non-zero integer. In [1], Adler asserted and proved the fact that $N(T_{\gamma,\mu})=2$. However I could not follow his proof. In this paper we shall assert and prove that $N(T_{\gamma,\mu})=3$. The method of the proof depends upon Adler's idea in [1].

2. Preliminaries. Let X be a half open unit interval $(0,1]$ equipped with the usual topology. Since X is homeomorphic to the circle group K by the mapping ρ of X onto K which is defined by $\rho(x)=\exp[2\pi i x]$, we may consider X as the circle group equipped with the normalized Haar measure. Let $H=X \times X$ be the topological product group of X and X equipped with the normalized Haar measure λ. We shall consider the following skew product measure-preserving transformation defined on H. Let $T_{\gamma,\alpha}$ denote the measure-preserving transformation with α-function which is defined by $T_{\gamma,\alpha}:(x,y)\to(x+\gamma, y+\alpha(x))$ (additions modulo 1) where γ is an irrational number and $\alpha(\cdot)$ a real-valued measurable function on X. Conditions for ergodicity of $T_{\gamma,\alpha}$ along with the proof that it is measure-preserving can be found in H. Anzai's paper [2]. Furthermore, two other results from [2] upon which the subsequent work depends are the following.

Proper value criterion. The value $\exp[2\pi i\xi]$ is a proper value of $T_{\gamma,\alpha}$ if and only if there exists an integer p and a real-valued measurable function $\theta(\cdot)$ on X such that

$$\xi-p\alpha(x)=\theta(x+\gamma)-\theta(x) \quad (\text{modulo } 1) \quad \text{a.e.}$$

Spatial isomorphism criterion. If S is an invertible measure-preserving transformation such that $ST_{\gamma,\alpha}S^{-1}=T_{\gamma,\beta}$ a.e. where $T_{\gamma,\alpha}$ and $T_{\gamma,\beta}$ are ergodic skew product transformations with α-function and β-function, respectively, then S has either the form

$$S:(x,y)\to(x+u, y+\theta(x))$$

(additions modulo 1) where u is a real constant and $\theta(\cdot)$ a real-valued measurable function on X such that

$$\beta(x+u)-\alpha(x)=\theta(x+\gamma)-\theta(x) \quad (\text{modulo } 1) \quad \text{a.e.}$$

or

$$S:(x,y)\to(x+u, -y+\theta(x))$$

(additions modulo 1) where u and $\theta(\cdot)$ now satisfy

$$\beta(x+u)+\alpha(x)=\theta(x+\gamma)-\theta(x) \quad (\text{molulo } 1) \quad \text{a.e.}$$

3. Generalized commuting properties. Let γ be an irrational number and $\alpha(\cdot)$ denote a real-valued measurable function on X of the form $\alpha: x\to\mu x+\delta$ where μ is a non-zero integer and δ a real constant. We shall restrict ourselves to the skew product transformation $T_{\gamma,\alpha}$ with the above α-function.

Theorem. *The generalized commuting order* $N(T_{\gamma,\alpha})=3$. *Furthermore* $C_0(T_{\gamma,\alpha})$, $C_1(T_{\gamma,\alpha})$, $C_2(T_{\gamma,\alpha})$, *and* $C_3(T_{\gamma,\alpha})$ *are subgroups of the group* $\mathfrak{G}$ *of all invertible measure-preserving transformations of* $(H,\mathfrak{M},\lambda)$ *where* $\mathfrak{M}$ *is the* σ*-field of all* λ*-measurable subsets of* H.

The theorem is established in a sequence of propositions.

Lemma. *If* T *is the invertible measure-preserving transformation on* H *which is defined by* $T:(x,y)\to(x+\gamma, y+\mu x+\delta)$ (*additions modulo* 1) *where* γ *is an irrational number,* μ *a non-zero integer, and* δ *a real constant, then* T *is totally ergodic and has quasi-discrete spectrum of order* 2.

The proof is not difficult, whence we omit it here (refer to [3]).

Proposition 1. $S\in C_1(T_{\gamma,\alpha})$, *i.e.,* $ST_{\gamma,\alpha}=T_{\gamma,\alpha}S$ *a.e. if and only if* S

almost everywhere is of the form

$$S:(x,y)\to\left(x+\frac{m\gamma+q}{\mu},\ y+mx+c\right)$$

(*additions modulo* 1) *where m is an integer,* $q=0,1,2,\cdots,$ *or* $|\mu|-1$, *and c a real constant.*

The proof is analogous to that of [1, Proposition 1, p. 9], whence we omit the details.

Let S be an invertible measure-preserving transformation on H such that $S^{\mu}=T_{\gamma,\alpha}$ a.e. Then S commutes with $T_{\gamma,\alpha}$ and so it almost everywhere must have the form

$$S:(x,y)\to\left(x+\frac{m\gamma+q}{\mu},\ y+mx+c\right)$$

(additions modulo 1), whence S^{μ} almost everywhere is of the form $S^{\mu}:(x,y)\to\left(x+m\gamma,\ y+\mu mx+\mu c+\frac{1}{2}[\mu-1]m[m\gamma+q]\right)$ (additions modulo 1). This together with $S^{\mu}=T_{\gamma,\alpha}$ implies $m\gamma=\gamma$ (modulo 1), whence $m=1$. Thus

$$\mu c+\frac{1}{2}(\mu-1)(\gamma+q)=\delta \quad \text{(modulo 1)}$$

i.e.

$$c=[2\delta+(1-\mu)(\gamma+q)+2q']/2\mu \quad \text{(modulo 1)} \tag{1}$$

where $q'=0,1,2,\cdots,$ or $|\mu|-1$. Conversely if S almost everywhere is of the form $S:(x,y)\to\left(x+\frac{\gamma+q}{\mu},\ y+x+c\right)$ (additions modulo 1) where c is defined by (1), then $S^{\mu}=T_{\gamma,\alpha}$ a.e.

Now it is easy to see that if S is a μth root of $T_{\gamma,\alpha}$ then the family of the transformations almost everywhere equal to one of the forms S^nR where n is an integer and $R:(x,y)\to\left(x+\frac{q}{\mu},\ y+c\right)$ in which $q=0,1,2,\cdots,$ or $|\mu|-1$ and c a real constant coincides with $C_1(T_{\gamma,\alpha})$.

Proposition 2. *$S\in C_2(T_{\gamma,\alpha})$ if and only if S almost everywhere is of the form*

$$S:(x,y)\to(\varepsilon x+u,\ y+kx+c)$$

(*additions modulo* 1) *where* $\varepsilon=1$ *or* -1, *k is an integer, and c a real constant.*

Proof. Let $S\in C_2(T_{\gamma,\alpha})$. Then $ST_{\gamma,\alpha}S^{-1}T_{\gamma,\alpha}^{-1}\in C_1(T_{\gamma,\alpha})$, whence

$$ST_{\gamma,\alpha}S^{-1}T_{\gamma,\alpha}^{-1}=U^nR \quad \text{a.e.}$$

where U is a μth root of $T_{\gamma,\alpha}$ and $R:(x,y)\to\left(x+\frac{q}{\mu},\ y+d\right)$ (additions modulo 1). Therefore

$$ST_{\gamma,\alpha}S^{-1}=U^{n+\mu}R \quad \text{a.e.}$$

The transformation on the right is the ergodic skew product transformation $T_{\frac{(n+\mu)\gamma+q}{\mu},\beta}$ with β-function which has the form $\beta: x \to [n+\mu]x + d'$ in which all the constants involved are lumped together in d'. Here we note that $n+\mu \neq 0$. This follows from the ergodicity of $T_{\frac{(n+\mu)\gamma+q}{\mu},\beta}$. By the proper value criterion $\exp[2\pi i\xi]$ is a proper value of $T_{\gamma,\alpha}$ if and only if there exists an integer p and a real-valued measurable function $\theta(\cdot)$ on X such that

$$\xi - p(\mu x+\delta) = \theta(x+\gamma) - \theta(x) \quad \text{(modulo 1)} \quad \text{a.e.}$$

This implies that $\exp[2\pi i\theta(\cdot)]$ is a generalized proper function of $T_\gamma: x \to x+\gamma$ (modulo 1) on X, whence the same argument as in the proof of [1, Proposition 1, p. 9] demonstrates that $\exp[2\pi i\theta(x)] = \exp[2\pi i(mx+c)]$ a.e. for some integer m and real constant c. Thus $\xi - p(\mu x+\delta) = m\gamma$ (modulo 1) a.e. and so $p=0$. It follows that $\{\exp[2\pi im\gamma] \mid m \text{ is an integer}\}$ is the proper values of $T_{\gamma,\alpha}$. The same argument as the above implies that $\left\{\exp\left[2\pi im\cdot\frac{(n+\mu)\gamma+q}{\mu}\right] \middle| m \text{ is an integer}\right\}$ is the proper values of $T_{\frac{(n+\mu)\gamma+q}{\mu},\beta}$. Since $T_{\gamma,\alpha}$ and $T_{\frac{(n+\mu)\gamma+q}{\mu},\beta}$ are spatially isomorphic the proper values of $T_{\gamma,\alpha}$ coincide with the proper values of $T_{\frac{(n+\mu)\gamma+q}{\mu},\beta}$ from which it follows that

$$q=0, \text{ and } (n+\mu)/\mu = 1 \text{ or } -1.$$

Let $(n+\mu)/\mu = 1$, i.e., $n=0$. Then $ST_{\gamma,\alpha}S^{-1}$ almost everywhere is of the form

$$ST_{\gamma,\alpha}S^{-1}: (x,y) \to (x+\gamma, y+\mu x+d')$$

(additions modulo 1). By the spatially isomorphism criterion S almost everywhere is of the form

(i) $S: (x,y) \to (x+u, y+\theta(x))$

(additions modulo 1) where

$$\mu(x+u)+d'-(\mu x+\delta) = \theta(x+\gamma)-\theta(x) \quad \text{(modulo 1)} \quad \text{a.e.}$$

or

(ii) $S: (x,y) \to (x+u, -y+\theta(x))$

(additions modulo 1) where

$$\mu(x+u)+d'+(\mu x+\delta) = \theta(x+\gamma)-\theta(x) \quad \text{(modulo 1)} \quad \text{a.e.}$$

In either case the argument of generalized proper functions assures that $\theta(x) = kx+c$ (modulo 1) a.e. for some integer k and real constant c, from which it follows that case (ii) is impossible.

Next let $(n+\mu)/\mu = -1$, i.e., $n = -2\mu$. Then $ST_{\gamma,\alpha}S^{-1}$ almost everywhere is of the form

$$ST_{\gamma,\alpha}S^{-1}: (x,y) \to (x-\gamma, y-\mu x+d')$$

(additions modulo 1). If Q denote the transformation on H which is defined by $(x, y)\to(-x, -y)$ then $QST_{\gamma,\alpha}S^{-1}Q^{-1}$ almost everywhere is of the form

$$QST_{\gamma,\alpha}S^{-1}Q^{-1}: (x, y)\to(x+\gamma, y-\mu x-d')$$

(additions modulo 1). Therefore the spatial isomorphism criterion can be applied to QS and we see that QS almost everywhere is of the form

(iii) $QS: (x, y)\to(x+u, y+\theta(x))$

(additions modulo 1) where $-\mu(x+u)-d'-(\mu x+\delta)=\theta(x+\gamma)-\theta(x)$ (modulo 1) a.e. or

(iv) $QS: (x, y)\to(x+u, -y+\theta(x))$

(additions modulo 1) where $-\mu(x+u)-d'+(\mu x+\delta)=\theta(x+\gamma)-\theta(x)$ (modulo 1) a.e.

The same argument used in the case $(n+\mu)/\mu=1$ demonstrates that case (iii) is impossible and that in case (iv) QS almost everywhere is of the form $QS: (x, y)\to(x+u, -y+kx+c)$ (additions modulo 1), whence S almost everywhere is of the form $S: (x, y)\to(-x-u, y-kx-c)$ (additions modulo 1).

Conversely if S almost everywhere is of the form

$$S: (x, y)\to(\varepsilon x+u, y+kx+c)$$

(additions modulo 1) then $ST_{\gamma,\alpha}S^{-1}T_{\gamma,\alpha}^{-1}$ almost everywhere is of the form

$$ST_{\gamma,\alpha}S^{-1}T_{\gamma,\alpha}^{-1}: (x, y)\to(x+[\varepsilon-1]\gamma, y+[\varepsilon-1]\mu x+c')$$

(additions modulo 1). Proposition 1 implies now $S\in C_2(T_{\gamma,\alpha})$. This completes the proof.

Proposition 3. *$S\in C_3(T_{\gamma,\alpha})$ if and only if S almost everywhere is of the form*

$$S: (x, y)\to(\varepsilon_1 x+u, \varepsilon_2 y+kx+c)$$

(additions modulo 1) where ε_1 and ε_2 equal 1 or -1, respectively, k is an integer, u and c are real constants.

It is easily seen that the same argument used in the proof of Proposition 2 can be applied in order to prove Proposition 3. Thus we omit the proof here.

Proposition 4. $C_4(T_{\gamma,\alpha})=C_3(T_{\gamma,\alpha})$, *i.e.*, $N(T_{\gamma,\alpha})=3$.

Proof. Let $S\in C_4(T_{\gamma,\alpha})$. Then $S_3=ST_{\gamma,\alpha}S^{-1}T_{\gamma,\alpha}^{-1}$ almost everywhere is of the form $S_3: (x, y)\to(\varepsilon_1 x+u, \varepsilon_2 y+kx+d)$ (additions modulo 1), whence $ST_{\gamma,\alpha}S^{-1}=S_3T_{\gamma,\alpha}$ almost everywhere is of the form

$$ST_{\gamma,\alpha}S^{-1}: (x, y)\to(\varepsilon_1 x+u_1, \varepsilon_2 y+k_1x+d_1)$$

(additions modulo 1) where k_1 is some integer and u_1, d_1 real constants. Thus $ST_{\gamma,\alpha}^2S^{-1}$ almost everywhere is of the form

$$ST_{\gamma,\alpha}^2S^{-1}: (x, y)\to(x+[\varepsilon_1+1]u_1, y+[\varepsilon_1+\varepsilon_2]k_1x+d_2)$$

(additions modulo 1) where d_2 is some real constant. Since $T_{\gamma,\alpha}$ is totally ergodic and has quasi-discrete spectrum of order 2, it follows that $\varepsilon_1+1\neq 0$ and $\varepsilon_1+\varepsilon_2\neq 0$, in other words, $\varepsilon_1=\varepsilon_2=1$. This together with Proposition 2 implies now that $S_3=ST_{\gamma,\alpha}S^{-1}T_{\gamma,\alpha}^{-1}$ belongs to $C_2(T_{\gamma,\alpha})$. Therefore S belongs to $C_3(T_{\gamma,\alpha})$. This completes the proof.

Remark. Let (M, Σ, m) be a non-atomic Lebesgue space (see [4]) with $m(M)=1$. Then it is known that if $T \in \mathfrak{G}$ is totally ergodic and has quasi-discrete spectrum of order 1 then $N(T)=2$ (see [1]). However I do not know whether if $T \in \mathfrak{G}$ is totally ergodic and has quasi-discrete spectrum of order 2 then $N(T)=3$.

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References

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