73. *On Infinitesimal Automorphisms of Siegel Domains*

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The aim of this note is to announce some theorems (Theorem 1–Theorem 4) concerning the Lie algebra $\mathfrak{g}$ of all infinitesimal automorphisms of a Siegel domain D of second kind. Theorems 3 and 4 enable us to calculate, in an algebraic manner, the Lie algebra $\mathfrak{g}$ on the basis of the Lie algebra $\mathfrak{g}_a$ of all infinitesimal affine automorphisms of D.

1. Let W^{-2} (resp. W^{-1}) be a real (resp. complex) vector space of finite dimension. We say that an open set V of W^{-2} is a convex cone in W^{-2} if it satisfies the following conditions:

1) $x+x'$, $\lambda x \in V$ for any $x, x' \in V$ and any real number $\lambda>0$,

2) V contains no entire straight lines.

Given a convex cone V in W^{-2}, we say that a mapping F of $W^{-1}\times W^{-1}$ to W_c^{-2} (=the complexification of W^{-2}) is a V-hermitian form on W^{-1} if it satisfies the following conditions:

1) F is hermitian, i.e., $F(u, u')$ is complex linear with respect to the variable u, and $\overline{F(u, u')}=F(u', u)$,

2) F is V-positive definite, i.e., $F(u, u) \in \bar{V}$ for any u, and $F(u, u) \neq 0$ for any $u\neq 0$, where $\bar{V}$ denotes the closure of V in W^{-2}.

Suppose that we are given a convex cone V in W^{-2} and a V-hermitian form F on W^{-1}. We put $\tilde{W}=W_c^{-2}+W^{-1}$ and denote by z^{-2} (resp. z^{-1}) the projection of $\tilde{W}$ onto W_c^{-2} (resp. onto W^{-1}). Furthermore we define a mapping Φ of $\tilde{W}$ to W^{-2} by

$$\Phi(p)=\operatorname{Im} z^{-2}(p)-F(z^{-1}(p), z^{-1}(p)) \qquad (p \in \tilde{W}).$$

Then the domain $D=\Phi^{-1}(V)$ (=the inverse image of V by Φ) of $\tilde{W}$ is called the Siegel domain of second kind associated with the cone V and the V-hermitian form F (Pyatetski-Shapiro [2]). Let S be the real submanifold of $\tilde{W}$ defined by $\Phi=0$, i.e., $S=\Phi^{-1}(0)$. Then [2] has asserted that S is just the Silov boundary of the domain D with respect to an appropriate ring of holomorphic functions on D.

2. Hereafter we assume that D is affine homogeneous, that is, the group of all affine transformations of $\tilde{W}$ leaving D invariant acts transitively on D. A holomorphic vector field on D is called an infinitesimal automorphism of D if it generates a one parameter group of automorphisms of D or equivalently if it is complete as a vector field.

An infinitesimal automorphism of D is called linear (resp. affine) if it is (extended to) an infinitesimal linear (resp. affine) transformation of $\tilde{W}$. (Under the identification that $\tilde{W}=T_p(\tilde{W})$ (=the tangent space to $\tilde{W}$ at any $p\in\tilde{W}$), an infinitesimal affine transformation of $\tilde{W}$ may be described as a mapping of the form: $\tilde{W}\ni p\rightarrow a+Ap\in\tilde{W}$, where $a\in\tilde{W}$ and A is an endomorphism of $\tilde{W}$.)

Theorem 1. *Every infinitesimal automorphism of D is extended to a holomorphic vector field which is defined on the whole $\tilde{W}$ and which is tangent to the Silov boundary S of D.*

Let E denote the infinitesimal linear transformation of $\tilde{W}$ defined by

$$E(p)=-2z^{-2}(p)-z^{-1}(p) \qquad (p\in\tilde{W})$$

Then we see that E is an infinitesimal linear automorphism of D.

Theorem 2. *Let $\mathfrak{g}$ be the Lie algebra of all infinitesimal automorphisms of D and, for any integer p, let $\mathfrak{g}^p$ be the subspace of $\mathfrak{g}$ consisting of all the elements X such that $[E,X]=pX$. Then we have:*

(1) $\mathfrak{g}=\sum_p\mathfrak{g}^p$ *(direct sum) and it is a graded Lie algebra.*

(2) $\mathfrak{g}^p=\{0\}$ $(p<-2)$, *and* $\mathfrak{g}_a=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0$ *is the Lie algebra of all infinitesimal affine automorphisms of D. More precisely, $\mathfrak{g}^0$ is the Lie algebra of all infinitesimal linear automorphisms of D, and $\mathfrak{m}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}$ is the Lie algebra of all infinitesimal "parallel translations" of D.*

(3) *$\mathfrak{g}$ being identified with a Lie algebra of holomorphic vector fields on $\tilde{W}$, the direct* sum $\sum_{p\geqq 0}\mathfrak{g}^p$ *is characterized as the isotropy algebra of $\mathfrak{g}$ at the origin 0 of $\tilde{W}$.*

(4) *Let p be any integer $\geqq 0$. Then the condition "$X\in\mathfrak{g}^p$, $[X,\mathfrak{m}]=\{0\}$" implies $X=0$.*

Remark. We first remark that the Lie algebra $\mathfrak{g}^0$ consists of all endomorphisms X of $\tilde{W}$ satisfying the following conditions (cf. [2]):

1) $XW^p\subset W^p$ $(p=-2,-1)$,

2) $XF(u,u')=F(Xu,u')+F(u,Xu')$,

3) X restricted to W^{-2} is an infinitesimal automorphism of the cone V.

Let $w^p\in W^p$ $(p=-2,-1)$ and put $w=w^{-2}+w^{-1}$. Define an infinitesimal affine transformation $s(w)$ of $\tilde{W}$ by

$$s(w)(p)=w^{-2}+2\sqrt{-1}\,F(z^{-1}(p),w^{-1})+w^{-1} \quad (p\in\tilde{W}).$$

Then we see that $s(w)$ is an infinitesimal affine automorphism of D, which has been called an infinitesimal parallel translation of D (cf. [2]). We remark that $\mathfrak{g}^p$ $(p=-2,-1)$ consists of all $s(w)$ $(w\in W^p)$ and that

$$[s(w),s(w')]=4s(\mathrm{Im}F(w,w')) \quad (w,w'\in W^{-1}),$$

$$[X,s(w)]=s(Xw) \quad (w\in W^{-2}+W^{-1}, X\in\mathfrak{g}^0).$$

3. Let us now construct a graded Lie algebra $\hat{\mathfrak{g}}=\sum_p \hat{\mathfrak{g}}^p$ satisfying the following conditions (cf. N. Tanaka [3], § 5):

1) $\sum_{p\leqq 0} \hat{\mathfrak{g}}^p=\sum_{p\leqq 0} \mathfrak{g}^p$ as graded Lie algebras,

2) Let p be any integer $\geqq 0$. Then the condition "$X\in \hat{\mathfrak{g}}^p$, $[X, \mathfrak{m}]=\{0\}$" implies $X=0$,

3) $\hat{\mathfrak{g}}$ is maximum among the graded Lie algebras satisfying conditions 1) and 2). More precisely, let $\mathfrak{f}=\sum_p \mathfrak{f}^p$ be any graded Lie algebra satisfying conditions 1) and 2). Then $\mathfrak{f}$ is imbedded in $\hat{\mathfrak{g}}$ as a graded subalgebra.

We put $\hat{\mathfrak{g}}^p=\hat{\mathfrak{g}}^p$ $(p<0)$. Since the condition "$X\in\mathfrak{g}^0, [X, \mathfrak{m}]=\{0\}$" implies $X=0$, we see that $\mathfrak{g}^0$ may be identified with a subspace of $\mathfrak{q}^0=\sum_{r<0}\mathrm{Hom}(\mathfrak{g}^r, \mathfrak{g}^r)\subset \mathrm{Hom}(\mathfrak{m}, \mathfrak{m})$. This being said, we have

$$[X^0(Y^r), Z^s]-[X^0(Z^s), Y^r]=X^0([Y^r, Z^s])$$

for all $Y^r\in\mathfrak{g}^r, Z^s\in\mathfrak{g}^s$ $(r, s<0)$. Let us define vector spaces $\hat{\mathfrak{g}}^p$ $(p\geqq 0)$ inductively as follows: First of all we define $\hat{\mathfrak{g}}^0$ as $\mathfrak{g}^0$. Suppose now that we have defined $\hat{\mathfrak{g}}^p$ $(0\leqq p<k)$ in such a way that $\hat{\mathfrak{g}}^p$ is a subspace of $\mathfrak{q}^p=\sum_{r<0}\mathrm{Hom}(\mathfrak{g}^r, \hat{\mathfrak{g}}^{r+p})\subset\mathrm{Hom}(\mathfrak{m}, \sum_{r<0}\hat{\mathfrak{g}}^{r+p})$. Then we define $\hat{\mathfrak{g}}^k$ to be the subspace of $\mathfrak{q}^k=\sum_{r<0}\mathrm{Hom}(\mathfrak{g}^r, \hat{\mathfrak{g}}^{r+k})$ which consists of all $X^k\in\mathfrak{q}^k$ satisfying the following equalities:

$$X^k(Y^r)(Z^s)-X^k(Z^s)(Y^r)=X^k([Y^r, Z^s])$$

for all $Y^r\in\mathfrak{g}^r, Z^s\in\mathfrak{g}^s$ $(r, s<0)$, where we put $X^k(Y^r)(Z^s)=[X^k(Y^r), Z^s]$ (if $r+k<0$) and $X^k(Z^s)(Y^r)=[X^k(Z^s), Y^r]$ (if $s+k<0$). Thus we have completed our inductive definition. We put $\hat{\mathfrak{g}}=\sum_p\hat{\mathfrak{g}}^p$. Then we see easily that there is a unique bracket operation $[\ ,\ ]$ in $\hat{\mathfrak{g}}$ such that $\hat{\mathfrak{g}}$ becomes a graded Lie algebra satisfying conditions 1) and 2) with respect to this bracket operation and such that $[X^k, Y^r]=X^k(Y^r)$ for all $X^k\in\hat{\mathfrak{g}}^k$, $Y^r\in\mathfrak{g}^r$ $(k\geqq 0, r<0)$. Moreover it is easy to see that the graded Lie algebra $\hat{\mathfrak{g}}$ thus obtained satisfies condition 3). This graded Lie algebra is called the prolongation of $\mathfrak{g}_a=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0$.

By Theorem 2, (4), we know that $\mathfrak{g}$ is a graded subalgebra of $\hat{\mathfrak{g}}$ in a natural manner.

Theorem 3. *Let $\mathfrak{g}=\sum_p\mathfrak{g}^p$ be the graded Lie algebra in Theorem 2 and let $\hat{\mathfrak{g}}=\sum_p\hat{\mathfrak{g}}^p$ be the prolongation of $\mathfrak{g}_a=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0$. For each $X\in\mathfrak{g}^0$, denote by* $\mathrm{Tr}(X)$ *the trace of X as an endomorphism of $\tilde{W}$. Then $\mathfrak{g}$ is a graded subalgebra of $\hat{\mathfrak{g}}$ and the subspaces $\mathfrak{g}^p\subset\hat{\mathfrak{g}}^p$ $(p>0)$ are inductively determined as follows:*

(1) $\mathfrak{g}^1=\hat{\mathfrak{g}}^1$.

(2) *$\mathfrak{g}^2$ consists of all $X\in\hat{\mathfrak{g}}^2$ such that* $\mathrm{Im}\,\mathrm{Tr}([X, Y])=0$ *for all $Y\in\mathfrak{g}^{-2}$.*

(3) *$\mathfrak{g}^3$ consists of all $X\in\hat{\mathfrak{g}}^3$ such that $[X, \mathfrak{g}^{-1}]\subset\mathfrak{g}^2$.*

(4) $\mathfrak{g}^4$ *consists of all* $X \in \hat{\mathfrak{g}}^4$ *such that* $[X, \mathfrak{g}^{-1}] \subset \mathfrak{g}^3$ *and* $\mathrm{Tr}([[X, Y], Y]) = 0$ *for all* $Y \in \mathfrak{g}^{-2}$.

(5) *For each* $k > 4$, $\mathfrak{g}^k$ *consists of all* $X \in \hat{\mathfrak{g}}^k$ *such that* $[X, \mathfrak{g}^{-2}] \subset \mathfrak{g}^{k-2}$ *and* $[X, \mathfrak{g}^{-1}] \subset \mathfrak{g}^{k-1}$.

Theorem 4. *Assume that* W^{-2} *is generated by the elements of the form* $F(u, u)$ $(u \in W^{-1})$, *or equivalently* $\mathfrak{g}^{-2} = [\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$. *Then we have* $\mathfrak{g} = \hat{\mathfrak{g}}$.

Kaneyuki-Sudo [1] has shown that the assumption in Theorem 4 is always satisfied if the Siegel domain D is symmetric and if each irreducible component of D is not of tube type.

References

[1] S. Kaneyuki and M. Sudo: On Silov boundaries of Siegel domains. J. Fac. Sci. Univ. of Tokyo, **15**, 131–146 (1968).

[2] I. I. Pyatetski-Shapiro: Geometry of classical domains and theory of automorphic functions, fizmatgiz (1961).

[3] N. Tanaka: On differential systems, graded Lie algebras and pseudo-groups (to appear).