134. *Propagation of Chaos for Certain Markov Processes of Jump Type with Nonlinear Generators. II*

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This is a continuation of the previous paper [3], and treats a generalization of Wild's sum for $\{H_p^t\}$ and the propagation of chaos for the nonlinear equation (1.1). All the notations are preserved; §§1, 2 and numbered formulas which are quoted here are in [3].

3. A generalization of Wild's sum. The expression (2.1) defining the linear semigroup $\{H_p^t\}$ associated with the equation (1.1) leads naturally to a generalization of Wild's sum [1] as will be explained here. Denote by τ^k, $k\geq 1$, the tree with only one branching point which is k-fold, and give a number j $(1\leq j\leq k)$ to each extreme point (or top) of the tree τ^k as in Fig. 1. We define the set T_n, $n\geq 1$, of trees with n extreme points and also the numbering to extreme points of each tree in T_n, inductively as follows.

i) $T_1=\{\tau^1\}$, $T_2=\{\tau^2\}$.

ii) $\tau\in T_n$, $n\geq 2$, is either a) $\tau=\tau^n$ or b) $\tau=(\tau', i, j)$ with $\tau'\in T_{n-j+1}$, $1\leq i\leq n-j+1$, $2\leq j\leq n$, where $(\tau'\ i,\ j)$ denotes the tree which is obtained by connecting τ^j at the i-th top of τ'. In particular, $(\tau^1, 1, n)$ is τ^n itself. In the case $\tau=(\tau', i, j)$, those extreme points of τ which are also extreme points of τ^j have the numbers $i, n-j+2, n-j+3, \cdots, n$, while other extreme points of τ have the same numbers as τ' (see Fig. 2).

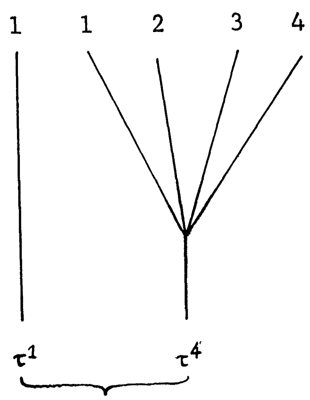

Fig. 1

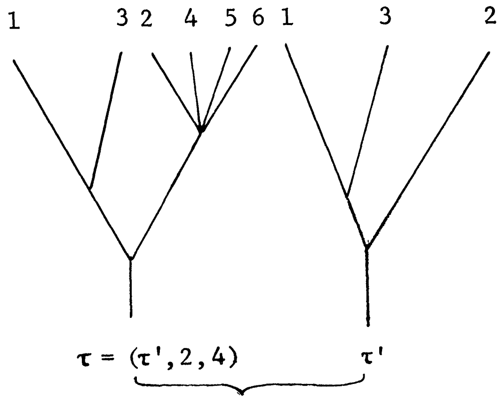

Fig. 2

Next, we set $T=\bigcup_{n=1}^{\infty} T_n$, $N(\tau)=n$ for $\tau\in T_n$, and $T_1'=T$,

$T'_k=\{\tau \in T;$ the first branching point is k-fold$\}$, $k \geq 2$.

For each $p=0, 1, \cdots, \tau \in T'_k, \varphi \in \Phi^k$, we define $\tau_p(\varphi) \in \Phi^{N(\tau)}$ as follows:

1) if $\tau=\tau^k$, $\tau_p(\varphi)=\varphi$

2) if $\tau=(\tau', i, j)$, $1 \leq i \leq m=N(\tau')$, $j \geq 2$, then

$$\tau_p(\varphi)=\begin{cases}\Pi_{j-p-1}(x_i, x_{m+p+1}, \cdots, x_{m+j-1}, \tau'_p(\varphi)), & j \geq p+2\\ 0, & j<p+2.\end{cases}$$

Let $\mathbf{M}_p=\{\tau(t), P^\tau_p(\cdot), \tau \in T\}$ be a minimal Markov process with state space T such that a particle starting at τ waits there during exponential holding time with expectation $\frac{1}{nq}$ $(n=N(\tau))$ and then jumps to the state (τ, i, j) with probability q_{j-p-1}/nq, $1 \leq i \leq n$, $j \geq p+2$. Then, the semigroup $\{H^t_p\}$ of §2 is expressed in terms of the Markov process $\mathbf{M}_p$. In fact, we can prove the following theorem, by considering the forward equation for $\mathbf{M}_p$ and the backward equation for the Markov process on $\mathbf{Q}$ determined by $\{\mathbf{H}^t_p\}$.

Theorem 2. *For each* $p \geq 0$, $\varphi \in \Phi^k$, $1 \leq k \leq n$,

$$(\mathbf{H}^t_p\boldsymbol{\varphi})_n=\sum_{\tau \in T'_k \cap T_n} P^{\tau^k}_p\{\tau(t)=\tau\}\tau_p(\varphi),$$

where $(\boldsymbol{\varphi})_n=\varphi$ *for* $n=k$, *and* $=0$ *for* $n \neq k$. *Therefore, under the assumption* (A),

$$H^t_p\hat{\varphi}=\theta_p\sum_{n=k}^{\infty}\sum_{\tau \in T'_k \cap T_n} P^{\tau^k}_p\{\tau(t)=\tau\}\tau_p(\varphi)$$

for $\hat{\varphi}=\theta_p\varphi$, $\varphi \in \Phi^k$.

As a corollary to this theorem, we can obtain similar formulas to Wild's sum [1] for the solution $u(t)$ of (1.1) and for the transition function $\{P_f(t, x, \Gamma)\}$. The first one was also obtained by S. Tanaka [2] by a different method. For each $\tau \in T$, $f_1, \cdots, f_n \in \mathcal{P}$ $(n=N(\tau))$, we define $\tau[f_1, \cdots, f_n] \in \mathcal{P}$ as follows: i) if $\tau=\tau^1$, $\tau[f]=f$, ii) if $\tau=\tau^n$ $(n \geq 2)$, then $\tau[f_1, \cdots, f_n]=\langle f_1 \otimes \cdots \otimes f_n, \Pi_{n-1}(x_1, x_2, \cdots, x_n, \cdot)\rangle$, and iii) if $\tau=(\tau', i, j)$, then

$$\tau[f_1, \cdots, f_n]=\tau'[\cdots, f_{i-1}, \tau^j[f_i, f_{n-j+2}, \cdots, f_n], f_{i+1}, \cdots, f_{n-j}].$$

Corollary 1. (a) *Let* $u(t)$ *be the solution of* (1.1). *Then,*

$$u(t)=\sum_{n=1}^{\infty}\sum_{\tau \in T_n} P^{\tau^1}_0\{\tau(t)=\tau\}\tau[\overbrace{f, \cdots, f}^{n}]$$

(b) $$P_f(t, x, \cdot)=\sum_{n=1}^{\infty}\sum_{\tau \in T_n} P^{\tau^1}_0\{\tau(t)=\tau\}\tau[\delta_x, \overbrace{f, \cdots, f}^{n-1}]$$

where δ_x *is the probability measure concentrated at* x.

4. Propagation of chaos. Let $D_{p,n}$, for each $p=0, 1, \cdots$, be a linear operator from $\hat{\Phi}^n_p$ into itself defined by

$$D_{p,n}\hat{\varphi}=\theta_p\sum_{N=1}^{n-1} n^{-N} \sum_{i, i_1, \cdots, i_N}^{(n)} A_N^{(x_{i_1}, \cdots, x_{i_N})}(x_i, \varphi)$$

where $\hat{\varphi}=\theta_p\varphi$, $\varphi \in \Phi^n$, and $\sum_{i, i_1, \cdots, i_N}^{(n)}$ is the same as in §1. Then, it turns out that $D_{p,n}$ is a bounded operator on the Banach space $\hat{\Phi}_p^n$, and hence there exists a semigroup $\{H_{p,n}^t\}$ on $\hat{\Phi}_p^n$ with generator $D_{p,n}$. On the other hand, the method of §2 can be adapted with some modifications to obtain an expression for $\{H_{p,n}^t\}$ which is similar to (2.1), and with the aid of this the following convergence theorem is proved.

Theorem 3. *Under the assumption* (A),

$$\lim_{n\to\infty} H_{p,n}^t \hat{\varphi} = H_p^t \hat{\varphi} \qquad \text{for } \hat{\varphi} \in \hat{\Phi}_p^\infty.$$

This theorem together with the multiplicative property of $\{H_0^t\}$ implies the following propagation of chaos.

Corollary 2. *Under the assumption* (A),

$$\lim_{n\to\infty} \langle u_n(t), \varphi \rangle = \langle u(t)^m, \varphi \rangle, \ \varphi \in \Phi^m,$$

where $u(t)$ is the solution of (1.2) *and in the left hand side φ is considered as a function on* $\mathbf{Q}^n$.

References

[1] E. Wild: On Boltzmann's equations in the kinetic theory of gases. Proc. Cambridge Philos. Soc., **47**, 602–609 (1951).

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[3] H. Tanaka: Propagation of chaos for certain Markov processes of jump type with nonlinear generators. I. Proc. Japan Acad., **45**, 449–452 (1969).