118. *On a General Form of the Weyl Criterion in the Theory of Asymptotic Distribution. II*

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V. Applications. 1. Let u_n $(n=1, 2, \cdots)$ be a sequence of real numbers. Then define the function f on $[0, \infty)$ as follows:

$$f(n)=u_n \qquad (n=1, 2, \cdots),$$
$$f(t)=f([t]+1) \qquad (t\not\equiv 0 \pmod 1).$$

Let $B(t)=[t]$ $(t\not\equiv 0 \pmod 1)$ and continuous on the left for every t. Then using the same notation as in IV we have

$$F_T(\xi)=\frac{1}{B(T)}\int_0^T \chi_{[0,\xi)}((f(t)))dB(t)$$

(where the integral is taken over the interval $[0, T)$)

$$=\frac{1}{[T]}\sum_{\substack{n=1\\ 0\le f(n)<\xi}}^{[T]} 1$$

Let the d.f. $F(\xi)$ equal to $\xi(0\le\xi\le 1)$, and $=0$ $(\xi\le 0)$ and $=1$ $(\xi\ge 1)$. Then $F_T(\xi)\xrightarrow{c}\xi$, as $T\to\infty$, if and only if for $k=1, 2, \cdots$

$$\lim_{T\to\infty}\frac{1}{[T]}\int_0^T \exp 2\pi i k f(t)d[T]$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp 2\pi i k u_n=\int_0^1 \exp 2\pi i k x dx=0,$$

or, Theorem 5 implies the Weyl criterion for the uniform distribution mod 1 of a sequence of real numbers. See [1].

2. Let u_n, $f(t)$, $B(t)$ and $F_T(\xi)$ be defined as in **1**. Let $F(\xi)$ be a d.f. with $F(\xi)=0$ $(0\le\xi<1)$ and $F(\xi)=1$ $(\xi>1)$. Suppose furthermore that $\Delta F(0)=\Delta F(1)=0$. Then

$$F_T(\xi)\xrightarrow{c}F(\xi), \qquad \text{as } T\to\infty,$$

if and only if

$$\lim_{T\to\infty}\frac{1}{B(T)}\int_0^T \exp 2\pi i k f(t)dB(t)=\lim_{T\to\infty}\frac{1}{[T]}\sum_{n=1}^{[T]}\exp 2\pi i k u_n$$
$$=\int_0^1 \exp 2\pi i k x dF(x).$$

Moreover

$$F_T(\xi)=\frac{1}{B(T)}\int_0^T \chi_{[0,\xi)}((f(t)))dB(t)=\frac{1}{[T]}\sum_{\substack{n=1\\ 0\le f(n)<\xi}}^{[T]} 1.$$

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In this case the sequence u_n $(n=1, 2, \cdots)$ (mod 1) is said to have the distribution function $F(\xi)$. See [10] where the case of the continuous d.f. $F(\xi)$ is treated.

3. Let $f(t)$ be a realvalued Borel measurable function defined on $[0, \infty)$. Set $f(t)=[f(t)]+(f(t))$. Let $B(t)=t(0\leq t<\infty)$. Define the class of d.f.

$$F_T(\xi)=\frac{1}{T}\int_0^T \chi_{[0,\xi)}((f(t)))dt(T>0),$$

$$F_T(\xi)=0\ (\xi\leq 0), \qquad F_T(\xi)=1\ (\xi>1).$$

Let $F(\xi)$ be defined as in 2. We have then that, as $T\to\infty$, $F_T(\xi)$ $\xrightarrow{c} F(\xi)$, the relative measure of the set

$$\{t: t\geq 0, 0\leq (f(t))<\xi\},$$

if and only if for $k=1, 2, \cdots$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \exp 2\pi ikf(t)dt=\int_0^1 \exp 2\pi ikxdF(x).$$

In this case $f(t)$ is said to have mod 1 the distribution function $F(\xi)$. See [11]. The case that $F(\xi)=\xi$ $(0\leq\xi\leq 1)$, or, that the function $f(t)$ is uniformly distributed mod 1 was already treated by Hermann Weyl. See [1].

4. Let $f(t)$ be defined as in **3**. Let $B(t)$ be a nondecreasing function defined on $[0, \infty)$ and continuous on the left. Let $F_T(\xi)$ be defined as in Theorem 5, and $F(\xi)$ as in 4. Suppose $\Delta F(0)=\Delta F(1)=0$. Then

$$F_T(\xi)\xrightarrow{c} F(\xi), \qquad \text{as } T\to\infty,$$

if and only if for $k=1, 2, \cdots$

$$\lim_{T\to\infty}\frac{1}{B(T)}\int_0^T \exp 2\pi ikf(t)dB(t)=\int_0^1 \exp 2\pi ikxdF(x).$$

In this case $f(t)$ is said to have the B-distribution function $F(\xi)$ mod 1. See [12].

5. Let a_n $(n=1, 2, \cdots)$ be a sequence of integers. Define the function $f(t)$ on $[0, \infty)$ as follows:

$f(n)=\dfrac{a_n(\text{mod } m)}{m}$, where a_n (mod m) is the number $j(0\leq j<m-1)$ such that $a_n\equiv j$ (mod m), $f(t)$ constant on $n\leq t<n+1$ and $=f([t])$, $n=1, 1, 2, \cdots$.

Define the d.f. F as follows:

$$F(\xi)=\frac{j+1}{m} \text{ if } \frac{j}{m}<x\leq\frac{j+1}{m}(j=0, 1, \cdots, m-1),$$

so that $\Delta F(0)>0$, $\Delta F(1)=0$.

Define the class of d.f.

$$F_T(\xi)=\frac{1}{[T]}\int_0^T \chi_{[0,\xi)}((f(t))d[t]=\frac{1}{[T]}\sum_{\substack{n=1\\0\le f(n)<\xi}}^{[T]} 1.$$

Let a and b be numbers with

$$\frac{j-1}{m}<a<\frac{j}{m}<b<\frac{j+1}{m}.$$

In this case ($\varDelta F(0)>0$) the Remarks 1 and 2 apply. We then have

$$F_T(b)-F_T(a)=\frac{1}{B(T)}\int_0^T \chi_{[\alpha,\beta)}(f(t))dB(t)$$

$$=\frac{1}{[T]}\sum_{\substack{n=1\\a\le f(n)<b}}^{[T]} 1=\frac{1}{[T]}\sum_{\substack{n=1\\f(n)\,(\mathrm{mod}\,m)=j/m \text{ or } a_n(\mathrm{mod}\,m)=j}}^{[T]} 1$$

$$\rightarrow \quad F(b)-F(a)=\frac{1}{m}, \quad \text{as } T\rightarrow\infty,$$

if and only if,

$$\lim_{T\to\infty}\frac{1}{B(T)}\int_0^T \exp 2\pi ikf(t)dB(t)=\lim_{T\to\infty}\frac{1}{T}\sum_{n=1}^{T}\exp 2\pi ika_n(\mathrm{mod}\ m)/m$$

$$=\lim_{T\to\infty}\frac{1}{T}\sum_{n=1}^{T}\exp 2\pi ika_n/m=\int_0^1 \exp 2\pi ikxdF(x)$$

(according to Theorem 5)

$$=\frac{1}{m}\sum_{j=0}^{m-1}\exp 2\pi ikj/m \ (k\not\equiv 0 \ (\mathrm{mod}\ m))=0,$$

and herewith we have proved the Niven-Uchiyama criterion for the uniform distribution mod m of a sequence of integers. See [6], [7].

6. Let u_n ($n=1, 2, \cdots$) be a sequence of real numbers. Define $f(t)$ as in Application 1.

Let $B(t)=B(t, x)$ be the function

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!},$$

where n is determined by $n<t\le n+1$ ($n=0, 1, \cdots$). Then for $x>0$ $B(t, x)$ is a nondecreasing function of t, continuous in t on the left. Define the class of d.f.

$$F_T(\xi)=F_T(\xi, x)=\frac{1}{B(T)}\int_0^T \chi_{[0,\xi)}((f(t))dB(t) \qquad (0\le\xi<1),$$

$F_T(\xi)=0$ ($\xi\le 0$), $F_T(\xi)=1$ ($\xi>1$). Let $F(\xi, x)$ be a d.f. with $F(\xi, x)=0$ ($\xi\le 0$), $=1$ ($\xi\ge 1$), and $\varDelta F(0)=\varDelta F(1)=0$. Furthermore it is assumed that $\lim_{x\to\infty} F(\xi, x)=F(\xi)$.

Then according to Theorem 5 we have:

$$F_T(\xi, x)\xrightarrow{c} F(\xi, x), \qquad \text{as } T\rightarrow\infty,$$

if and only if for $k=1, 2, \cdots$

$$\lim_{T\to\infty}\frac{1}{B(T)}\int_0^T \exp 2\pi ikf(t)dB(t)$$

$$=\lim_{N\to\infty}\frac{1}{1+\dfrac{x}{1!}+\cdots+\dfrac{x^n}{n!}}\sum_{n=1}^{N}\frac{e^{2\pi iku_n}\cdot x^n}{n!}$$

$$=e^{-x}\sum_{n=1}^{\infty}\frac{e^{2\pi iku_n}\cdot x^n}{n!}=\int_0^1 e^{2\pi ik\xi}dF(\xi, x).$$

The last equality implies

$$\lim_{x\to\infty} e^{-x}\sum_{n=1}^{\infty}\frac{e^{2\pi iku_n}\cdot x^n}{n!}=\int_0^1 e^{2\pi ik\xi}dF(\xi),$$

in other words: the sequence u_n is mod 1 Borel distributed to the distribution function $F(\xi)$. See [13].

Final Remark. In concluding we observe that [14] contains a chapter on weak convergence in metric spaces. If S is a metric space, and if $\mathcal{S}$ is the class of Borel sets in S, if P and P_n ($n=1, 2, \cdots$) are probability measures, then (there) P_n is said to converge weakly to P if

$$\int_S fdP_n\to\int_S fdP$$

for every bounded continuous real function f on S. Billingsley applies his results to the case of weak convergence on the circle and the torus and acquires in this way Weyl's criterion (see Application 1 of this paper) and its 2-dimensional extension.

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