174. *Elliptic Modular Surfaces. I*

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Introduction. Let B denote an elliptic surface over a non-singular algebraic curve Δ having a global holomorphic section o. We denote by J and G the functional and homological invariants of B, and by $\mathcal{F}(J, G)$ the family of elliptic surfaces over Δ with the same functional and homological invariants as B. We refer to Kodaira [1] for the general theory of elliptic surfaces. The family $\mathcal{F}(J, G)$, modulo suitable equivalence, is parametrized by the cohomology group $H^1(\Delta, \Omega(B^\#))$ (or by $H^1(\Delta, \Omega(B_0^\#))$), where $B^\#$ (or $B_0^\#$) denotes the group scheme over Δ associated with B (or the connected component of the identity section o in $B^\#$) and where $\Omega(B^\#)$ (or $\Omega(B_0^\#)$) denotes the sheaf of germs of holomorphic sections of $B^\#$ (or $B_0^\#$) over Δ. Moreover the torsion elements in the group $H^1(\Delta, \Omega(B^\#))$ correspond to algebraic surfaces in the family $\mathcal{F}(J, G)$. Now Kodaira raised the question whether or not algebraic surfaces are dense in the family $\mathcal{F}(J, G)$, which has motivated our present work.

In this (and the forthcoming) paper, we consider the special case where B, Δ, J and G are defined in terms of a torsion-free subgroup Γ of finite index of the homogeneous modular group $SL(2, \boldsymbol{Z})$ (see Section 2 for the definition). We write, if necessary, $B_\Gamma, \Delta_\Gamma, \cdots$ for $B, \Delta, \cdots$ to specify the group Γ. The base curve Δ_Γ of B_Γ is the compact Riemann surface associated with the Fuchsian group Γ acting on the upper half plane. The elliptic surface B_Γ will be called the *elliptic modular surface* attached to Γ. The main results can be stated as follows.

Theorem. *Let B denote an elliptic modular surface with the base curve Δ. Then*

(i) *B has only a finite number of global sections over Δ.*

(ii) *The group $H^1(\Delta, \Omega(B_0^\#))$ is isomorphic to a product of a complex torus and a finite group.*

The complex torus in question is an analogue of Shimura's abelian varieties attached to cusp forms of even weights [3]; here it is related to cusp forms of weight 3. We do not know whether or not this complex torus has an structure of an abelian variety. As an immediate consequence of the theorem, we get a partial answer to Kodaira's question:

Corollary. *Algebraic surfaces are dense in the family $\mathcal{F}(J, G)$ containing an elliptic modular surface.*

The contents of the paper are as follows. In Section 1, we recall an analytic proof of Ogg-Šafarevič's formula for an elliptic curve over a function field; it has been known for some time to Kodaira and Tate, but we state it here for our later use. In Section 2, we define, for each subgroup Γ of $SL(2, \boldsymbol{Z})$ mentioned above, an "elliptic modular surface"; we give a basic example at the end of the section. The proof of the above theorem and the various remarks will be given in Part II, which will appear shortly.

We would like to remark that elliptic modular surfaces provide good examples for the arithmetic theory of surfaces. For instance, the following could be checked for the elliptic modular surface X for $\Gamma(4)$, the principal congruence subgroup of level 4:

1) X is defined over $\boldsymbol{Q}(\sqrt{-1})$.
2) Riemann hypothesis for the reduction of X mod $\mathfrak{p}$ for almost all prime $\mathfrak{p}$ of $\boldsymbol{Q}(\sqrt{-1})$.
3) For each $p \equiv 1\ (4)$, let $p = a^2 + (2b)^2$ with positive integers a, b. Then, assuming Conjecture C of Tate [5], the order of the Brauer group is

$$|\operatorname{Br}(X \bmod \mathfrak{p}/\boldsymbol{F}_p)| = b^2. \qquad (p = \mathfrak{p}\bar{\mathfrak{p}}).$$

(Note that our X is not rational, nor product of two curves.)

In this paper we shall state the results omitting most of the proofs. Detailed accounts will be published elsewhere.

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1. Preliminaries. We use the same notations $B, \Delta, \cdots$ as in the biginning part of the introduction. Throughout the paper, we assume that the fibres of an elliptic surface do not contain exceptional curves. Also we assume that the functional invariant J is not a constant. (Hence, the homological invariant G is non-trivial). Let g denote the genus of the algebraic curve Δ and let t and t_1 denote, resp., the total number of singular fibres of B over Δ and the number of singular fibres of type $I_b\ (b \geq 1)$ (see [1], Section 6). Then the cohomology groups $H^i(\Delta, G)$ of Δ with coefficients in the sheaf G can be described as follows:

Lemma 1.1. *The group $H^1(\Delta, G)$ is finitely generated and of rank $4g-4+2t-t_1$. $H^2(\Delta, G)$ is a finite group and $H^i(\Delta, G)$ for $i \neq 1, 2$ vanishes.*

We denote by $\mathfrak{f}$ the normal bundle to (the image of) the identity section o in the surface B; $\mathfrak{f}$ is a line bundle over Δ of degree $-(p_a+1)$, p_a being the arithmetic genus of B. There is a fundamental exact sequence ([1], Section 11):

$$(*) \qquad 0 \longrightarrow G \xrightarrow{i} \mathcal{O}(\mathfrak{f}) \xrightarrow{h} \Omega(B_0^\#) \longrightarrow 0,$$

where $\mathcal{O}(\mathfrak{f})$ is the sheaf of germs of holomorphic sections of $\mathfrak{f}$. It induces the exact sequence of cohomology groups:

$$(**) \qquad 0 \longrightarrow H^0(\Delta, \Omega(B_0^\#)) \longrightarrow H^1(\Delta, G) \xrightarrow{i^*} H^1(\Delta, \mathcal{O}(\mathfrak{f}))$$
$$\xrightarrow{h^*} H^1(\Delta, \Omega(B_0^\#)) \longrightarrow H^2(\Delta, G) \longrightarrow 0.$$

The group of sections $H^0(\Delta, \Omega(B_0^\#))$ (or $H^0(\Delta, \Omega(B^\#))$) is finitely generated by the functional analogue of Mordell-Weil theorem. We call r the rank of the group $H^0(\Delta, \Omega(B_0^\#))$ and r' that of the image $i^*H^1(\Delta, G)$ in $H^1(\Delta, \mathcal{O}(\mathfrak{f}))$. From the exact sequence $(**)$ it is clear that r' is equal to the largest number n such that the torsion subgroup of $H^1(\Delta, \Omega(B_0^\#))$ contains a subgroup (of finite index) isomorphic to $(\boldsymbol{Q}/\boldsymbol{Z})^n$. Moreover we have

Theorem 1.2. $r + r' = 4g - 4 + 2t - t_1$.

This theorem is known as the formula of Ogg-Šafarevič, which has been proved for abelian varieties over function fields with arbitrary characteristic (cf. [2]).

Now let b_2 and ρ be the second Betti number and the Picard number of the algebraic elliptic surface B. They can be computed in terms of singular fibres, the genus g of Δ and the rank r of the group of sections. Then Theorem 1.2 implies the following result, which is also known in the abstract setting ([2]):

Theorem 1.3. $r' = b_2 - \rho$.

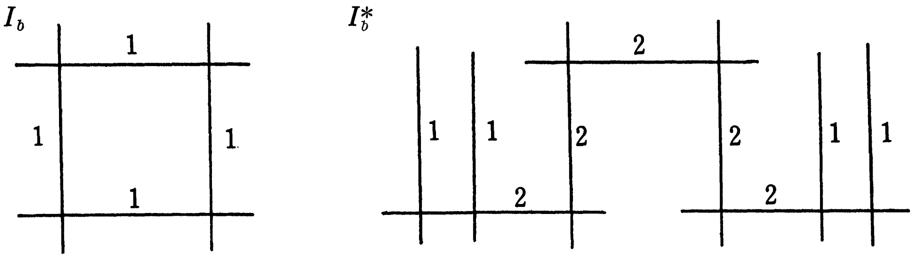
Corollary 1.4. *$r' \geq 2p_g$, p_g = the geometric genus of B.*

2. Definition of elliptic modular surfaces. Let Γ denote a torsion-free subgroup of finite index of the homogeneous modular group $SL(2, \boldsymbol{Z})$. The group Γ acts on the upper half plane $\mathfrak{H}$ in the usual manner and the quotient $\Gamma\backslash\mathfrak{H}$, together with a finite number of cusps, forms a compact Riemann surface, say Δ. Putting $\Delta' = \Gamma\backslash\mathfrak{H}$, we identify the fundamental group $\pi_1(\Delta')$ of Δ' with the Fuchsian group Γ and thus get a representation of $\pi_1(\Delta')$ in $SL(2, \boldsymbol{Z})$: $\pi_1(\Delta') = \Gamma \rightarrow SL(2, \boldsymbol{Z})$. Such a representation determines a sheaf G over Δ, locally constant of fibre $\boldsymbol{Z}^2$ over Δ'. On the other hand, the inclusion $\Gamma \subset SL(2, \boldsymbol{Z})$ defines a canonical holomorphic mapping of $\Delta' = \Gamma\backslash\mathfrak{H}$ onto $SL(2, \boldsymbol{Z})\backslash\mathfrak{H}$; the latter can be identified with the complex plane $\boldsymbol{C}$ by means of the ordinary elliptic modular function j. Thus we get a holomorphic function on Δ', which can be extended to a meromorphic function, say J, on Δ with poles on $\Delta - \Delta'$.

We can apply to this situation Kodaira's construction of elliptic surfaces ([1], Section 8); there exists a (non-singular algebraic) elliptic surface B over Δ with a global section having J and G as its functional and homological invariants. This elliptic surface will be called the *elliptic modular surface* attached to the Fuchsian group Γ. We write, if necessary, B_Γ, Δ_Γ, $\cdots$ for B, Δ, $\cdots$ to specify the group Γ. B has singular fibres over cusps of Γ (i.e. over points of $\Delta-\Delta'$); for a cusp x, the singular fibre over x is of type I_b or I_b^* ($b\geq 1$) according as the cusp x is of the first kind or of the second kind. We recall that a cusp x is of the first or of the second kind according as the stabilizer in Γ of (a representative of) x has a generator which is conjugate in $SL(2, \boldsymbol{Z})$ to

$$\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}-1 & -b\\ 0 & -1\end{pmatrix}, \qquad b>0.$$

Recall also that singular fibres of type I_b or I_b^* are described as follows:



(b components) ($b+5$ components)

Each line represents a non-singular rational curve, and the integer attached to a line indicates the multiplicity.

Example 2.1. Let $\Gamma(N)$ denote the principal congruence subgroup of level N. For $N\geq 3$, the group $\Gamma(N)$ is torsion-free and all cusps are of the first kind ($t=t_1, t_2=0$). The genus $g(N)$ and the number $t(N)$ of cusps are given by:

$$g(N)=1+(N-6)\,\mu(N)/12N, \qquad t(N)=\mu(N)/N,$$

where

$$\mu(N)=\frac{1}{2}N^3\prod_{p\mid N}(1-p^{-2}).$$

Let $B(N)$ denote the elliptic modular surface attached to $\Gamma(N)$. The surface $B(N)$ is a compactification of the universal family of elliptic curves with level N structure, the base being (the compactification of) the moduli variety. We call $B(N)$ the *elliptic modular surface for level N*. All singular fibres of $B(N)$ are of type I_N and the Picard number $\rho(N)$ is given by $\rho(N)=2+(N-1)\mu(N)/N$, while the geometric genus is $p_g(N)=(N-3)\mu(N)/6N$. By Theorem 1.3, the second Betti number $b_2(N)$ of $B(N)$ is equal to $\rho(N)+2p_g(N)$. It is interesting to note the asymptotic behavior:

$$\lim_{N\to\infty}\rho(N)/b_2(N)=3/4, \qquad \lim_{N\to\infty}p_g(N)/b_2(N)=1/8.$$

References

[1] K. Kodaira: On compact analytic surfaces. II-III. Ann. of Math., **77**, 563–626; **78**, 1–40 (1963).

[2] A. P. Ogg: Cohomology of abelian varieties over function fields. Ann. of Math., **76**, 185–212 (1962).

[3] G. Shimura: Sur les intégrales attachées aux formes automorphes. Jour. Math. Soc. Japan, **11**, 291–311 (1959).

[4] ——: On Dirichlet series and abelian varieties attached to automorphic forms. Ann. of Math., **72**, 237–294 (1962).

[5] J. Tate: On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. Seminaire Bourbaki, no **306**, 1–26 (1965–66).