188. On Certain Mixed Problem for Hyperbolic Equations of Higher Order. III

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1. Introduction. Let $\boldsymbol{R}_+^n$ be the open half space $\{(x, y); x>0, y \in \boldsymbol{R}^{n-1}\}$. We consider the mixed problem $(P, B_j; j=1, \cdots, l)$, briefly (P, B_j), for hyperbolic equations of order m in $(0, T)\times \boldsymbol{R}_+^n$ $(0<T<\infty)$:

$$(P(D_t, D_x, D_y)u)(t, x, y)=f(t, x, y) \quad \text{in } (0, T)\times \boldsymbol{R}_+^n,$$
$$(B_j(D_t, D_x, D_y)u)(t, 0, y)=0 \ (j=1, \cdots, l) \quad \text{in } (0, T)\times \boldsymbol{R}^{n-1},$$
$$(D_t^k u)(0, x, y)=0 \ (k=0, 1, \cdots, m-1) \quad \text{in } \boldsymbol{R}_+^n,$$

where $D_t=\frac{\partial}{\partial t}$, $D_x=-i\frac{\partial}{\partial x}$, $D_y=\left(-i\frac{\partial}{\partial y_1}, \cdots, -i\frac{\partial}{\partial y_{n-1}}\right)$ and $i=\sqrt{-1}$.

The purpose of this paper is to determine the necessary and sufficient conditions for L^2-well-posedness in the following sense.

Definition. *The mixed problem (P, B_j) is L^2-well-posed if and only if there exist constants T and T' with $0<T'\leq T$ which satisfy the following condition:*

For every $f\in H^1((-\infty, T)\times \boldsymbol{R}_+^n)$ with $f=0$ $(t<0)$ the mixed problem (P, B_j) has a unique solution $u \in H^m((0, T')\times \boldsymbol{R}_+^n)$ so that

$$\sum_{k=0}^{m-1}\int_0^{T'} \|(D_t^k u)(t, \cdot, \cdot)\|_{m-k-1}^2 dt \leq C\int_0^T \|f(t, \cdot, \cdot)\|_0^2 dt,$$

where a constant C depends only on T.

In § 2 we give certain necessary and sufficient conditions for L^2-well-posedness (Theorem 1) and investigate zeros of the Lopatinskii's determinant under L^2-well-posedness (Theorem 2).

In T. Shirota and K. Asano [5] it has been shown by semi-group method that the mixed problem $(P, D_x^{2j-1}; j=1, \cdots, l)$ $(m=2l)$ is *well posed in the L^2-sense*[1] if $P(D)=P(D_t, D_x, D_y)$ does not contain the terms of odd order relative to D_x. As one of the applications of Theorems 1 and 2 we show that, in the case of constant coefficients, the above condition for $P(D)$ is necessary to be *well posed in the L^2-sense* for the above mixed problem. This assertion is found in Theorem 4 in § 3.

The details and other results will be published elsewhere.

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1) This term means that in our definition one changes the inequality into the energy inequality.

2. Necessary and sufficient conditions for L^2-well-poseness. Let $P(D)$ and $B_j(D)$ $(j=1, \cdots, l)$ be homogenuous differential operators of order m and $m_j (m_j<m)$ with constant coefficients respectively. We assume that $P(D)$ is strongly hyperbolic relative to t-direction and the hyperplane $x=0$ is non-characteristic for $P(D)$. Then it is easily seen that the number $l(m-l)$ of the roots $\lambda_j^+(\tau, \sigma)$ $(j=1, \cdots, l)$ $(\lambda_k^-(\tau, \sigma)$ $(k=1, \cdots, m-l))$ in λ of the characteristic equation $P(\tau, \lambda, \sigma)=0$ located in the upper (lower) half λ-plane is constant for any (τ, σ) with $\mathrm{Re}\,\tau>0$ and $\sigma \in \boldsymbol{R}^{n-1}$ respectively.

Throughout this paper we use the following Fourier-Laplace transforms and norms.

$$\hat{u}(\tau, \lambda, \sigma)=\int_0^\infty dt\int_0^\infty dx\int_{\boldsymbol{R}^{n-1}} e^{-\tau t-i\lambda x-i\sigma y}u(t, x, y)dy,$$

$$\hat{u}(\tau, x, \sigma)=\int_0^\infty dt\int_{\boldsymbol{R}^{n-1}} e^{-\tau t-i\sigma y}u(t, x, y)dy,$$

$$|||u(t, \cdot, \cdot)|||_k^2=\sum_{j=0}^{k}\|(D_t^j u)(t, \cdot, \cdot)\|_{k-j}^2,$$

$$|||\hat{u}(\tau, \cdot, \cdot)|||_k^2=\sum_{j=0}^{k}\int_{\boldsymbol{R}^{n-1}}(|\tau|^2+|\sigma|^2)^{k-j}d\sigma\int_0^\infty|(D_x^j\hat{u})(\tau, x, \sigma)|^2dx \ (\mathrm{Re}\,\tau\geq\gamma>0),$$

where $\sigma y=\sigma_1 y_1+\cdots+\sigma_{n-1}y_{n-1}$, $|\sigma|^2=\sigma_1^2+\cdots+\sigma_{n-1}^2$, γ is arbitrarily fixed and $\|\ \|_h$ is the norm in Sobolev space $H^h(\boldsymbol{R}_+^n)$ $(h=0, 1, \cdots)$.

We define the Lopatinskii's determinant $R(\tau, \sigma)$ as follows:

$$B(\tau, \sigma)=\det(B_1(\tau, \lambda_k^+(\tau, \sigma), \sigma), \cdots, B_l(\tau, \lambda_k^+(\tau, \sigma), \sigma);\ k\downarrow 1, \cdots, l),$$

$$R(\tau, \sigma)=B(\tau, \sigma)/\prod_{1\leq j<k\leq l}(\lambda_k^+(\tau, \sigma)-\lambda_j^+(\tau, \sigma)).$$

Note that $R(\tau, \sigma)$ is analytic in $\mathrm{Re}\,\tau>0$ and real analytic in $\boldsymbol{R}^{n-1}$. Let V be the set $\{(\tau, \sigma);\ R(\tau, \sigma)=0,\ \mathrm{Re}\,\tau>0,\ \sigma\in\boldsymbol{R}^{n-1}\}$ and $S(\tau)$ the analytic variety $V\cap\{(\tau, \sigma);\ \sigma\in\boldsymbol{R}^{n-1}\}$. Then we have $\alpha V=V$ and $\alpha S(\tau)=S(\alpha\tau)$ for every $\alpha>0$.

Applying now the Fourier-Laplace transform to the equations in the problem (P, B_j) we obtain the boundary value problem $(\hat{P}, \hat{B}_j)$ of the ordinary differential equations depending parameters (τ, σ) with $\mathrm{Re}\,\tau>0$ and $\sigma\in\boldsymbol{R}^{n-1}$:

$$(P(\tau, D_x, \sigma)\hat{u})(\tau, x, \sigma)=\hat{f}(\tau, x, \sigma) \qquad \text{in } \boldsymbol{R}_+^1,$$

$$(B_j(\tau, D_x, \sigma)\hat{u})(\tau, 0, \sigma)=0 \quad (j=1, \cdots, l).$$

Let $R_j(\tau, x, \sigma)$ be the determinant replacing the j-column in $R(\tau, \sigma)$ by the transposed vector of $(e^{ix\lambda_1^+(\tau,\sigma)}, \cdots, e^{ix\lambda_l^+(\tau,\sigma)})$ and $\Gamma=\Gamma(\tau, \sigma)$ a closed Jordan curve in the lower half λ-plane enclosing all the roots $\lambda_k^-(\tau, \sigma)$ $(k=1, \cdots, m-l)$. If $R(\tau, \sigma)$ is not zero for some (τ, σ) with $\mathrm{Re}\,\tau>0$ and $\sigma\in\boldsymbol{R}^{n-1}$, then it is well known that for every $\hat{f}(\tau, \cdot, \sigma)\in C_0^\infty(\boldsymbol{R}_+^1)$ the boundary value problem $(\hat{P}, \hat{B}_j)$ has a unique solution $\hat{u}(\tau, \cdot, \sigma)\in C^\infty(\bar{\boldsymbol{R}}_+^1)$, which is written in the form:

$$\hat{u}(\tau, x, \sigma)=\frac{1}{2\pi}\int_0^\infty G_1(x, s, \tau, \sigma)\hat{f}(\tau, s, \sigma)ds+\frac{1}{2\pi}\int_0^\infty G_2(x, s, \tau, \sigma)\hat{f}(\tau, s, \sigma)ds,$$

where $G_1(x, s, \sigma)=\int_\Gamma \frac{e^{i(x-s)\lambda}}{P(\tau, \lambda, \sigma)}d\lambda,$

$$G_2(x, s, \tau, \sigma)=-\sum_{j=1}^{l}\frac{R_j(\tau, x, \sigma)}{R(\tau, \sigma)}\int_\Gamma \frac{B_j(\tau, \lambda, \sigma)}{P(\tau, \lambda, \sigma)}e^{-is\lambda}d\lambda.$$

Let Σ_+ be the set $\{(\tau', \sigma'); |\tau'|^2+|\sigma'|^2=1, \mathrm{Re}\,\tau'>0, \sigma'\in \boldsymbol{R}^{n-1}\}$ and $\bar{\Sigma}_+$ its closure. Set $V'=V\cap\Sigma_+$. When A is a subset of Σ_+ we denote the complement of A in Σ_+ by A^c. Then we have the following

Theorem 1. *If $S(\tau)$ is not the whole space $\boldsymbol{R}^{n-1}$ for every τ with* $\mathrm{Re}\,\tau>0$, *then the mixed problem (P, B_j) is L^2-well-posed if and only if the following condition* (I) *is satisfied:*

For every $(\tau_0', \sigma_0')\in(\bar{\Sigma}_+-\Sigma_+)\cup V'$ there exist a neighbourhood $U(\tau_0', \sigma_0')$ and a constant $C(\tau_0', \sigma_0')$ such that for any $(\tau', \sigma')\in U(\tau_0', \sigma_0')\cap\Sigma_+\cap V'^c$

$$\text{(I)}\qquad \|(D_x^kG_2)(x, s, \tau, \sigma)\|_{\mathcal{L}(L^2(s>0), L^2(x>0))}\leq\frac{C(\tau_0', \sigma_0')}{\mathrm{Re}\,\tau'}$$

$$(k=0, 1, \cdots, m-1),$$

where $\|\ \|_{\mathcal{L}(L^2(s>0), L^2(x>0))}$ is the operator norm from $L^2(s>0)$ to $L^2(x>0)$.

To prove Theorem 1 we need the following lemmas. Hereafter we denote various positive constants by C.

Lemma 1. *If a polynomial $P(\tau, \xi)$ of degree m is strongly hyperbolic relative to τ, then we have for any (τ, ξ) with* $\mathrm{Re}\,\tau>0$ *and* $\xi\in\boldsymbol{R}^n$

$$|P(\tau, \xi)|^2\geq C(\mathrm{Re}\,\tau)^2(|\tau|^2+|\xi|^2)^{m-1}.$$

Lemma 2. *If the assumption in Theorem* 1 *and the condition* (I) *are satisfied, then for every $(\tau, \sigma)\notin V$ and $f\in H^{k+1}((-\infty, \infty)\times\boldsymbol{R}_+^n)$ $(k=0, 1, \cdots)$ with $f=0$ $(t<0)$ the boundary value problem $(\hat{P}, \hat{B}_j)$ has a unique solution $\hat{u}(\tau, \cdot, \sigma)\in H^{m+k}(\boldsymbol{R}_+^1)$ so that*

$(\mathrm{Re}\,\tau)^2|||\hat{u}(\tau, \cdot, \cdot)|||_{m-1+k}^2\leq C|||\hat{f}(\tau, \cdot, \cdot)|||_k^2$ *for any τ with* $\mathrm{Re}\,\tau\geq\gamma>0$, *where γ is arbitrarily fixed and a constant C depends only on γ.*

Lemma 3. *If the assumption in Theorem* 1 *and the condition* (I) *are satisfied, then for every $a>0$ and $f\in H^{k+1}((-\infty, \infty)\times\boldsymbol{R}_+^n)$ $(k=0, 1, \cdots)$ with $f=0$ $(t<0)$ the mixed problem (P, B_j) (taking $T=\infty$) has a unique solution u which satisfies $e^{-at}u\in H^{m+k}((0, \infty)\times\boldsymbol{R}_+^n)$ and the following estimate*

$$\int_0^\infty e^{-2at}|||u(t, \cdot, \cdot)|||_{m-1+k}^2dt\leq\frac{C}{a^2}\int_0^\infty e^{-2at}|||f(t, \cdot, \cdot)|||_k^2dt,$$

where a constant C does not depend on u, f and a.

The following lemma is used in proof of necessity of Theorem 1.

Lemma 4. *Let f be a function in $L^2((-\infty, \infty)\times\boldsymbol{R}_+^n)$ whose support is contained in $(0, T)\times\boldsymbol{R}_+^n$ and u a function satisfying $e^{-at}u\in H^m((0, \infty)\times\boldsymbol{R}_+^n)$ for some $a>0$ and $(D_t^ku)(0, x, y)=0$ $(k=0, 1, \cdots, m-1)$ in $\boldsymbol{R}_+^n$. If*

$$\int_0^\infty e^{-2at}|||u(t,\cdot,\cdot)|||_{m-1}^2 dt \le C\int_0^\infty |||f(t,\cdot,\cdot)|||_0^2 dt,$$

then we have

$$|||\hat{u}(\tau,\cdot,\cdot)|||_{m-1}^2 \le C_0 C\int_{-\infty}^\infty |||\hat{f}(a+i\eta,\cdot,\cdot)|||_0^2 d\eta \quad \textit{for any } \tau \textit{ with } \operatorname{Re}\tau=a,$$

where the constant C_0 depends on a and the support of f.

Next we state the following theorem which shows that $S(\tau)$ must be the cone surface with its vertex at the origin in $\boldsymbol{R}^{n-1}$.

Theorem 2. *Suppose that the hyperplane $x=0$ is non-characteristic for $B_j(D)$ $(j=1,\cdots,l)$ and $m_1<\cdots<m_l$. If the mixed problem (P,B_j) is L^2-well-posed, then the varieties $S(\tau)$ don't depend on τ with* $\operatorname{Re}\tau>0$.

By Theorem 2 and the theory of characters of unitary group [6] we obtain

Corollary. *Under the same assumptions in Theorem 2, if $B_j(D)$ does not contain the terms relative to D_t and the mixed problem (P,B_j) is L^2-well-posed, the $S(\tau)$ is empty for any τ with* $\operatorname{Re}\tau>0$.

3. Applications. First we describe necessary and sufficient conditions for L^2-well-posedness by the terms of reflection coefficients. To define reflection coefficients, for every $(\tau_0',\sigma_0')\in\bar{\Sigma}_+-\bar{\Sigma}_+$ we rearrange the roots $\lambda_j^+(\tau',\sigma')$ $(j=1,\cdots,l)$ in a sufficiently small neighbourhood $U(\tau_0',\sigma_0')\cap\bar{\Sigma}_+$ such that $\lambda_{j_1}^+(\tau_0',\sigma_0')=\cdots=\lambda_{j_2-1}^+(\tau_0',\sigma_0')$ $(j_1=1),\cdots,$ $\lambda_{j_q}^+(\tau_0',\sigma_0')=\cdots=\lambda_{j_{q+1}-1}^+(\tau_0',\sigma_0')$ $(j_{q+1}-1=l)$. Then we define reflection coefficients $C_{k,j}(\tau',\lambda,\sigma')$ $(k=1,\cdots,q;\ j=j_k,\cdots,j_{k+1}-1)$ by the equality

$$\sum_{j=1}^{l}\frac{R_j(\tau',x,\sigma')}{R(\tau',\sigma')}B_j(\tau',\lambda,\sigma')=\sum_{k=1}^{q}\sum_{j=j_k}^{j_{k+1}-1}C_{k,j}(\tau',\lambda,\sigma')\gamma_{k,j}(\tau',x,\sigma'),$$

where $\gamma_{k,j_k}(\tau',x,\sigma')=e^{ix\lambda_{j_k}^+(\tau',\sigma')}$,

$$\gamma_{k,j}(\tau',x,\sigma')=x^{j-j_k}\int_0^1 d\theta_1\cdots\int_0^1\theta_1^{j-j_k-1}\cdots\theta_{j-j_k-1}e^{ixg_j(\tau',\sigma';\theta)}d\theta_{j-j_k},$$

$$g_j(\tau',\sigma';\theta)=\lambda_{j_k}^+(\tau',\sigma')+(\lambda_{j_k+1}^+(\tau',\sigma')-\lambda_{j_k}^+(\tau',\sigma'))\theta_1+\cdots$$
$$+(\lambda_j^+(\tau',\sigma')-\lambda_{j-1}^+(\tau',\sigma'))\theta_1\cdots\theta_{j-j_k}\quad(j_k<j<j_{k+1}).$$

The following condition is introduced by S. Agmon [1].

Condition (♯). *The multiplicity of a real root $\lambda(\tau,\sigma)$ in λ of the characteristic equation $P(\tau,\lambda,\sigma)=0$ is at most double for every (τ,σ) with* $\operatorname{Re}\tau=0$ *and $\sigma\in\boldsymbol{R}^{n-1}$.*

Then we have the following

Theorem 3. *Suppose the condition (♯). If $S(\tau)$ is not the whole space $\boldsymbol{R}^{n-1}$ for every τ with* $\operatorname{Re}\tau>0$, *then the mixed problem (P,B_j) is L^2-well-posed if and only if the following condition* (II) *is satisfied*:

(II) *For every $(\tau_0',\sigma_0')\in(\bar{\Sigma}_+-\Sigma_+)\cup V'$ there exist a neighbourhood $U(\tau_0',\sigma_0')$ and a constant $C(\tau_0',\sigma_0')$ such that for any (τ',σ') $\in U(\tau_0',\sigma_0')\cap\Sigma_+\cap V'^c$*

$$\left\| \int_{\Gamma} \frac{C_{k,j}(\tau', \lambda, \sigma')}{P(\tau', \lambda, \sigma')} e^{-is\lambda} d\lambda \right\|_{L^2(s>0)} \leq C(\tau_0', \sigma_0') \frac{|\mathrm{Im}\, \lambda_j^+(\tau', \sigma')|^{\frac{1}{2}}}{\mathrm{Re}\, \tau'},$$

$$(k=1, \cdots, q;\ j=j_k, \cdots, j_{k+1}-1).$$

From Theorem 3 we obtain the following

Theorem 4. *Let $P(D)$ and $Q(D)$ be homogenuuous differential operators, which don't contain the terms of odd order relative to D_x, of order $2l$ and $2l-1$ with constant coefficients respectively. If $P(D)$ satisfies the condition (♯), then the mixed problem $(P(D)+\varepsilon D_x Q(D), D_x^{2j-1};\ j=1, \cdots, l)$ is not well posed in the L^2-sense for a sufficiently small ε with certain fixed sign.*

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