6. *Applications of the Theorem Giving the Necessary and Sufficient Condition for the Normality of Product Spaces*

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We in this note intend to re-prove two well known results by means of Theorem in [1] furnishing the condition stated in the title. The first of the results suggests an explanation of Theorem from another point of view, and the second can be verified somewhat more shortly than the original proof.

Spaces in this note are Hausdorff, and notations and terminologies in [1] are used without referring.

Proposition 1 (C.H. Dowker [3]). *Let I be the real closed unit interval. If $X\times I$ is normal, then X is countably paracompact.*

Proof. It suffices ([3]) to show that for any decreasing sequence $\{F_i\}$ of closed sets of X with vacuous intersection we can find a sequence $\{G_i\}$ of open sets with vacuous intersection such that $F_i\subset G_i$ for all i. Let us denote $y_i=1/i$ for $i\neq 0$ and put

$$\begin{aligned} F_y&=F_i &&\text{for } y=y_i,\\ F_y&=\emptyset &&\text{otherwise};\\ K_y&=X &&\text{for } y=0,\\ K_y&=\emptyset &&\text{otherwise.}\end{aligned}$$

Then we have for any $b\in I$

$$\limsup_b F_y\cap\limsup_b K_y=\emptyset$$

(we here use the assumption $\cap F_i=\emptyset$ for $b=0$), and, by Theorem in [1], there are families $\{G_y\subset X;\ y\in I\}$ and $\{H_y\subset X;\ y\in I\}$ with the properties

$$G_y\cap H_y=\emptyset,$$
$$G_b^0\supset c\text{-}\limsup_b G_y\supset F_b,$$
$$c\text{-}\limsup_b H_y\supset K_b.$$

Since $c\text{-}\limsup_0 H_y\supset X$, there is, for any $x\in X$, $V\in\mathfrak{N}_0$ such that

$$x\in\Big(\bigcap_{y\in V}H_y\Big)^0\subset H_y$$

for any $y\in V$. Take $y_i\in V$, then $x\in H_{y_i}$ and $x\notin G_{y_i}$. Consequently, we have

$$\bigcap_i G_{y_i}=\emptyset,$$
$$G_{y_i}^0\supset F_{y_i}=F_i.$$

Remark. We can also interprete the property $P(X, Y)$ in Definition 2 in [1] as follows. We define the property

$P_3(X, Y)$: Given two maps g and h of X into the set 2^Y of all subsets of Y with

$$(**) \qquad \limsup_a g(x) \cap \limsup_a h(x) = \emptyset$$

for any $a \in X$, then there is a map f of X into 2^Y which *separates* g and h, i.e., satisfies the following condition. For any $x \in X$, $Y - f(x)$ can be represented as the union of disjoint subsets C_x and D_x of Y such that

$$c\text{-}\limsup_a C_x \supset g(a)$$

and

$$c\text{-}\limsup_a D_x \supset h(a)$$

for each $a \in X$.

$$f'(a) = \limsup_a (Y - C_x) \cap \limsup_a (Y - D_x)$$

also separates g and h.

$$F = \bigcup_{x \in X} (x, f'(x))$$

is closed in $X \times Y$. In fact, in general, we have

$$\bigcup_{x \in X} (x, L_x \cap M_x) = \Big[\bigcup_{x \in X} (x, L_x)\Big] \cap \Big[\bigcup_{x \in X} (x, M_x)\Big],$$

and $\bigcup_{z \in X} (z, \limsup_z N_x)$ is closed by Proposition 2 and Corollary 2 to it in [1]. So, by Corollary 1 to Proposition 2 in [1], we have

$$f'(a) = \limsup_a f'(x)$$

for any $a \in X$.

When $P(X, Y)$ is satisfied and maps g and h with $(**)$ are given, we consider $A_x = g(x)$ and $B_x = h(x)$, then the condition $(**)$ is the condition $(*)$ in Definition 2 in [1], and there are separators $\{G_x\}$ and $\{H_x\}$ of $\{A_x\}$ and $\{B_x\}$. The map

$$f(a) = \mathcal{C}(c\text{-}\limsup_a G_x) \cap \mathcal{C}(c\text{-}\limsup_a H_x)$$

separates g and h, where we consider $c\text{-}\limsup_a G_x$ and $c\text{-}\limsup_a H_x$ as C_a and D_a in the definition of $P_3(X, Y)$ respectively; therefore, $P(X, Y)$ implies $P_3(X, Y)$.

Conversely, when $P_3(X, Y)$ is satisfied and $\{A_x\}$ and $\{B_x\}$ are given together with the condition $(*)$ in Definition 2 in [1], we put $g(x) = A_x$ and $h(x) = B_x$, then there is a separating map f of g and h. Separators of $\{A_x\}$ and $\{B_x\}$ are $G_a = c\text{-}\limsup_a C_x$ and $H_a = c\text{-}\limsup_a D_x$, $a \in X$ (cf. Corollary 2 to Proposition 2 in [1]).

Now, let us consider a lower semi-continuous real map g, $0 \leqq g \leqq 1$, on X and an upper semi-continuous real map h, $0 \leqq h \leqq 1$, on X with $h(x) < g(x)$ for all $x \in X$. Take a point $a \in X$ and select numbers b_1 and

b_2 with $h(a)<b_1<b_2<g(a)$, then there is $U_0(\in \mathfrak{N}_a)$ such that $h(x)<b_1$ and $g(x)>b_2$ for any $x \in U_0$, i.e.,

$$\bigcup_{x \in U_0} [0, h(x)] \subset [0, b_1),$$

$$\bigcup_{x \in U_0} [g(x), 1] \subset (b_2, 1],$$

$$\left(\bigcap_{U \in \mathfrak{N}_a} \overline{\bigcup_{x \in U} [0, h(x)]}\right) \cap \left(\bigcap_{U \in \mathfrak{N}_a} \overline{\bigcup_{x \in U} [g(x), 1]}\right) = \emptyset,$$

which is the condition (**) above for $h'(x)=[0, h(x)]$ and $g'(x)=[g(x), 1]$. So, if the property $P_3(X, I)$, $I=[0,1]$, is satisfied in this case, then there is a separating map f of g' and h'. Put

$$f'(a)=\limsup_a (I-C_x) \cap \limsup_a (I-D_x),$$

where C_x and D_x correspond to those in the definition of $P_3(X, Y)$, then f' also separates g' and h', and

$$F=\bigcup_{x \in X} (x, f'(x))$$

is closed in $X \times I$ and upper semi-continuous at X (Definition 1 in [2]) because I is compact (Corollary 1 to Proposition 4 and Proposition 3 in [2]). The upper semi-continuity of F is nothing but the continuity of f' if f' is one-point-valued. Such an f' is constructed in [3] directly from the countable paracompactness of X.

For the next proposition we need two definitions given in [4].

Definition 1. Let Ω be a non-empty set. For any two sequences $\alpha=(\alpha_1, \alpha_2, \cdots)$ and $\beta=(\beta_1, \beta_2, \cdots)$ of elements from Ω we define a distance function $d(\alpha, \beta)$ as follows.

$$d(\alpha, \beta)=1/k \quad \text{if} \quad \alpha_i=\beta_i \quad \text{for} \quad i<k \quad \text{and} \quad \alpha_k \neq \beta_k,$$
$$d(\alpha, \beta)=0 \quad \text{if} \quad \alpha_i=\beta_i \quad \text{for} \quad \mathrm{i}=1, 2, \cdots.$$

The set of all sequences of elements from Ω equipped with d is a metric space called a *Baire space* $N(\Omega)$.

Definition 2. Let $\mathfrak{m}$ be a cardinal number $\geqq 1$. We shall say that a topological space X is a $P(\mathfrak{m})$-*space* if for a set Ω of power $\mathfrak{m}$ and for any family $\{F(\alpha_1, \cdots, \alpha_i); \alpha_1, \cdots, \alpha_i \in \Omega, i=1, 2, \cdots\}$ of closed subsets of X such that $F(\alpha_1, \cdots, \alpha_i) \supset F(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$ for $\alpha_1, \cdots, \cdots, \alpha_i, \alpha_{i+1} \in \Omega$, $i=1, 2, \cdots$, there is a family $\{G(\alpha_1, \cdots, \alpha_i); \alpha_1, \cdots, \cdots, \alpha_i \in \Omega, i=1, 2, \cdots\}$ of open subsets of X satisfying the two conditions:

$$G(\alpha_1, \cdots, \alpha_i) \supset F(\alpha_1, \cdots, \alpha_i) \quad \text{for } \alpha_1, \cdots, \alpha_i \in \Omega, \; i=1, 2, \cdots;$$

$$\bigcap_{i=1}^{\infty} G(\alpha_1, \cdots, \alpha_i)=\emptyset \quad \text{for any sequence } \{\alpha_i\} \text{ with } \bigcap_{i=1}^{\infty} F(\alpha_1, \cdots, \alpha_i)=\emptyset.$$

The next is the half of one of main theorems in [4].

Proposition 2 (K. Morita [4, Lemma 4.5]). *Let X be a topological* *space and Ω a set of power $\mathfrak{m}+1$, where $\mathfrak{m}\geqq 2$. Suppose that the product space $X\times S$ is normal for any subspace S of the Baire space $N(\Omega)$. Then X is a normal $P(\mathfrak{m})$-space.*

Proof. Select a point $\omega\in\Omega$ and put $\Omega'=\Omega-\{\omega\}$. Let $\{F(\alpha_1,\cdots,\cdots,\alpha_i);\ \alpha_1,\cdots,\alpha_i\in\Omega',\ i=1,2,\cdots\}$ be a family of closed subsets of X such that

$$F(\alpha_1,\cdots,\alpha_i)\supset F(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$$

for $\alpha_1,\cdots,\alpha_i\in\Omega'$, $i=1,2,\cdots$. Put

$$S'=\left\{(\alpha_1,\alpha_2,\cdots);\bigcap_{i=1}^{\infty}F(\alpha_1,\cdots,\alpha_i)=\emptyset\right\},$$

and to any point $\alpha=(\alpha_1,\alpha_2,\cdots)$ of S' and to any i we correspond the point $\alpha(i)=(\alpha_1,\cdots,\alpha_i,\omega_{i+1},\omega_{i+2},\cdots)$ of $N(\Omega)$, where $\omega_{i+1}=\omega_{i+2}=\cdots=\omega$, and we put

$$S''=\{\alpha(i);\ \alpha\in S',\ i=1,2,\cdots\},$$
$$S=S'\cup S''.$$

Let us further put

$$A_\xi=F(\alpha_1,\cdots,\alpha_i)\qquad\text{for } \xi=\alpha(i)\in S'',$$
$$A_\xi=\emptyset\qquad\text{for } \xi\in S';$$
$$B_\xi=\emptyset\qquad\text{for } \xi\in S'',$$
$$B_\xi=X\qquad\text{for } \xi\in S'.$$

Then we have

$$\limsup_{\beta} A_\xi\cap\limsup_{\beta} B_\xi=\emptyset$$

for any $\beta\in S$. In fact, if $\beta\in S'$ and $\beta=(\beta_1,\beta_2,\cdots)$, then (cf. Proposition 1 in [1])

$$\bigcap_n\overline{\bigcup_{\xi\in V(\beta,1/n)\cap S}A_\xi}=\bigcap_n\overline{\bigcup_{\xi\in V(\beta,1/n)\cap S''}A_\xi}$$
$$=\bigcap_n\overline{\bigcup_{i\geqq n}F(\beta_1,\cdots,\beta_i)}$$
$$=\bigcap_n F(\beta_1,\cdots,\beta_n)=\emptyset,$$

where $V(\beta,1/n)=\{\xi\in N(\Omega);\ d(\beta,\xi)<1/n\}$; if $\beta\in S''$ and $\beta=(\beta_1,\cdots,\cdots,\beta_{i_0},\omega_{i_0+1},\omega_{i_0+2},\cdots)$, then

$$\bigcup_{\xi\in V(\beta,1/n)\cap S}B_\xi=\emptyset$$

for any $n>i_0$ because $d(\beta,\xi)<1/n$ and $\xi\in S$ imply $\xi\in S''(\beta_i\neq\omega_j$ for any i and j and $V(\beta,1/n)\cap S$ consists of only $\beta)$.

Consequently, there are by Theorem in [1] separators $\{G_\xi\}$ and $\{H_\xi\}$ of $\{A_\xi\}$ and $\{B_\xi\}$. For any $\alpha\in S'$ we have $c\text{-}\limsup_{\alpha} H_\xi\supset B_\alpha=X$, and there are, for any $x\in X$, a natural number n_0 and a neighborhood U of x such that

$$U\subset\bigcap_{\xi\in V(\alpha,1/n_0)\cap S}H_\xi\subset\bigcap_{\xi\in V(\alpha,1/n_0)\cap S''}H_\xi.$$

Since $G_\xi\cap H_\xi=\emptyset$, we have

$$U\cap\left(\bigcup_{\xi\in V(\alpha,1/n_0)\cap S''}G_\xi\right)=\emptyset,$$
$$x\notin\overline{\bigcup_{\xi\in V(\alpha,1/n_0)\cap S''}G_\xi}.$$

Since $x \in X$ is arbitrary, we have

$$\bigcap_n \overline{\bigcup_{\xi \in V(\alpha, 1/n) \cap S''} G_\xi} = \emptyset.$$

$\{G^0_{\alpha(i)} = G(\alpha_1, \cdots, \alpha_i);\, i = 1, 2, \cdots\}$ is the desired open family for $\alpha \in S'$ because

$$F(\alpha_1, \cdots, \alpha_i) = A_{\alpha(i)} \subset c\text{-}\limsup_{\alpha(i)} G_\xi \subset G^0_{\alpha(i)} \subset \bigcup_{\xi \in V(\alpha, 1/i) \cap S''} G_\xi.$$

References

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